Solving the EEG inverse problem

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Outline

1. Inverse source reconstruction

2. (Blind) source separation
Electroencephalography (EEG)

Cellular (primary) currents due to synchronous firing of large populations of equally spatially coaligned neurons are accompanied by extracellular return (secondary) currents measurable as extracranial electric potentials by EEG.
Volume conduction: attenuation and spatial smearing

Primary current generator (dipole)

Resulting EEG scalp potential: ~5-10 μV
Volume conduction: attenuation and spatial smearing

Primary current generator (dipole)

Resulting EEG scalp potential: ~5-10 μV

Artifacts: ~100 μV
Volume conduction: superposition of activity

Primary current generator (dipole)

Resulting EEG scalp potential: ~5-10 μV
Volume conduction: superposition of activity

Primary current generator (dipole)

Resulting EEG scalp potential: \(~5-10 \, \mu V\)
Volume conduction: superposition of activity

Primary current generator (dipole)

Resulting EEG scalp potential: ~5-10 μV
Volume conduction: difficulties caused by

Primary current generator (dipole)

Resulting EEG scalp potential: ~5-10 μV

Difficulties for data analysis:
- Low SNR (small effect sizes, high p values)
- Localization/interpretation
Illustration: sensor-space analysis

Assume there is a brain area modulated by, e.g., the experimental condition.

\[ r = 0.71 \]
Illustration: sensor-space analysis

Assume there is a brain area modulated by, e.g., the experimental condition.

Due to contributions from other brain areas + noise, we will observe lower correlations and distorted correlation patterns in the EEG.

$r = 0.71$

$r = 0.27$
Model for EEG data

\[
x(t) = \sum_{u_i \in \mathcal{B}} L_i \mathbf{j}_i(t) + \epsilon(t) = L \mathbf{j}(t) + \epsilon(t)
\]

\[\mathcal{B} : \text{discretized brain}\]
Model for EEG data

\[ x(t) = \sum_{u_i \in B} L_i j_i(t) + \epsilon(t) = Lj(t) + \epsilon(t) \]

\( B \) : discretized brain

EEG scalp potential \( x(t) \in \mathbb{R}^M \) at \( M \) electrodes is a function of
Model for EEG data

\[ \mathbf{x}(t) = \sum_{\mathbf{u}_i \in \mathcal{B}} \mathbf{L}_i \mathbf{j}_i(t) + \epsilon(t) = \mathbf{Lj}(t) + \epsilon(t) \quad \mathcal{B} : \text{discretized brain} \]

EEG scalp potential \( \mathbf{x}(t) \in \mathbb{R}^M \) at \( M \) electrodes is a function of

\[ \mathbf{j}_i(t) \in \mathbb{R}^3 : \text{primary current at brain location } \mathbf{u}_i \]
Model for EEG data

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x(t) = \sum_{u_i \in B} L_i j_i(t) + \epsilon(t) = L j(t) + \epsilon(t)
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\[ j_i(t) \in \mathbb{R}^3 \text{ : primary current at brain location } u_i \]

\[ L_i \in \mathbb{R}^{M \times 3} \text{ : mapping describing the propagation of secondary currents to sensors for unit primary currents at } u_i \]
Model for EEG data

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\[ L \in \mathbb{R}^{M \times 3N} \quad \text{: lead field} \]
Model for EEG data

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\( L \in \mathbb{R}^{M \times 3N} \) : lead field (forward mapping for \( N \) brain locations)

\( j(t) \in \mathbb{R}^{3N} \) : primary current density
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\( j(t) \in \mathbb{R}^{3N} \) : primary current density

\( \epsilon(t) \in \mathbb{R}^M \) : electrical activity of no interest (sensor noise, artifacts)
The lead field (forward mapping)

$L$ depends on

- Conductivities of brain/skull/skin etc.
- Head geometry obtained from structural MRI
- Electrode positions (3D scanner)

Figure from Litvak et al., 2011
The lead field (forward mapping)

\( \mathbf{L} \) depends on

- Conductivities of brain/skull/skin etc.
- Head geometry obtained from structural MRI
- Electrode positions (3D scanner)

\( \mathbf{L} \) is evaluated at \( N \gg M \) brain locations \( \mathbf{u}_i \) in 3D volume or on cortical surface.

Figure from Litvak et al., 2011
The lead field (forward mapping)

\[ L \in \mathbb{R}^{M \times 3N} \], pre-calculated

Columns:

Rows:
The current density

\[ j(t) \in \mathbb{R}^{3N}, \]  
\text{to be estimated from } L \text{ and } x(t) \]

\[ t = 1 \]

Vectorfield, plotting magnitudes here.
The current density

\[ \mathbf{j}(t) \in \mathbb{R}^{3N} \], to be estimated from \( \mathbf{L} \) and \( \mathbf{x}(t) \)

\[ t = 2 \]

Vectorfield, plotting magnitudes here.
The current density

\[ j(t) \in \mathbb{R}^{3N}, \text{ to be estimated from } L \text{ and } x(t) \]

\[ t = 3 \]

Vectorfield, plotting magnitudes here.
The Inverse Problem

\[ x(t) = L j(t) + \epsilon(t) \]

We would like to invert the mapping \( L \) to obtain the current sources \( j(t) \).
The Inverse Problem

\[ x(t) = Lj(t) + \epsilon(t) \]

We would like to invert the mapping \( L \) to obtain the current sources \( j(t) \).

Potential benefits:
- Increase in SNR
- Localization/interpretation
The Inverse Problem

\[ x(t) = Lj(t) + \epsilon(t) \]

However, the inverse problem (estimating \( j \) from \( x \)) has infinitely many solutions.
The Inverse Problem

\[ \mathbf{x}(t) = \mathbf{L} \mathbf{j}(t) + \mathbf{\epsilon}(t) \]

However, the inverse problem (estimating \( \mathbf{j} \) from \( \mathbf{x} \)) has infinitely many solutions.
The Inverse Problem

\[ \mathbf{x}(t) = \mathbf{Lj}(t) + \mathbf{\epsilon}(t) \]

However, the inverse problem (estimating \( j \) from \( x \)) has infinitely many solutions.

Solving the inverse problem = selecting the sources that best match prior expectations (assumptions), while explaining the data.
Inverse methods

<table>
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<th>Method</th>
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<td>MCE</td>
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<tr>
<td>WMNE</td>
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<td>Laura</td>
<td>Dipole Modeling</td>
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<td>Electra</td>
<td>Multipole Modeling</td>
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<tr>
<td>WROP</td>
<td>MUSIC/RAP-MUSIC</td>
</tr>
<tr>
<td>S-FLEX</td>
<td>DCM</td>
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<tr>
<td>Champagne</td>
<td></td>
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</tbody>
</table>

Every method performs well if its specific assumptions are met.

No method can perform well in all realistic situations.
Maximum-likelihood and maximum a-posteriori estimation

\[ x(t) = Lj(t) + \epsilon(t) \]

Assuming the noise \( \epsilon(t) \) is Gaussian distributed with covariance \( Q \), the maximum-likelihood approach to estimating the source current density is

\[
\hat{j}_{ML}(t) = \arg \min_{j(t)} l(j(t)) \quad \text{with} \quad l(j(t)) = \| x(t) - Lj(t) \|_{Q^{-1}}^2.
\]
Maximum-likelihood and maximum a-posteriori estimation

\[
\mathbf{x}(t) = \mathbf{L}\mathbf{j}(t) + \mathbf{\epsilon}(t)
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Assuming the noise \( \mathbf{\epsilon}(t) \) is Gaussian distributed with covariance \( \mathbf{Q} \), the maximum-likelihood approach to estimating the source current density is

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\hat{\mathbf{j}}_{\text{ML}}(t) = \arg\min_{\mathbf{j}(t)} l(\mathbf{j}(t)) \quad \text{with} \quad l(\mathbf{j}(t)) = \| \mathbf{x}(t) - \mathbf{L}\mathbf{j}(t) \|^2_{\mathbf{Q}^{-1}}.
\]

However, since \( N \gg M \) (the system \( \mathbf{x} = \mathbf{L}\mathbf{j} \) is underdetermined),

\[
l(\mathbf{j}(t)) \quad \text{is zero for infinitely many choices of} \quad \mathbf{j}(t).
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However, since \( N \gg M \) (the system \( x = Lj \) is underdetermined), \( l(j(t)) \) is zero for infinitely many choices of \( j(t) \).

Need to impose additional penalty/constraint \( g(j(t)) \) on the sources.

**Maximum-a-posteriori estimate:**

\[ \hat{j}_{MAP}(t) = \arg\min_{j(t)} l(j(t)) + \lambda g(j(t)) \]
Maximum-likelihood and maximum a-posteriori estimation

\[ \mathbf{x}(t) = \mathbf{Lj}(t) + \mathbf{\epsilon}(t) \]

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Spatial constraints

Since $L$ links source activity to brain locations, constraints on the spatial structure of the current density can be imposed.
Spatial constraints: smoothness

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**Smoothness**

- Assumption: neighboring voxels show similar activity
- Examples: (weighted) minimum norm estimate, LORETA

[Jeffs et al., 1987; Pascual-Marqui et al., 1994]
Spatial constraints: smoothness

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Smoothness

- Assumption: neighboring voxels show similar activity
- Examples: (weighted) minimum norm estimate, LORETA

\[ g(j(t)) = \| \Gamma j(t) \|^2 \]

- Technically: $L_2$-norm penalty

\[
\hat{j}(t) = \left( L^T L + \lambda \Gamma^T \Gamma \right)^{-1} L^T x(t)
\]

- $P$ is precomputable $\Rightarrow$ very efficient
Spatial constraints: sparsity

**Sparsity**

- Assumption: only a small part of the brain is active during task
- E.g., minimum current estimate (MCE), FOCUSS

[Matsuura et al., 1995; Gorodnitsky et al., 1995]
Spatial constraints: sparsity

Sparsity

- Assumption: only a small part of the brain is active during task
- E.g., minimum current estimate (MCE), FOCUSS

\[ g(j(t)) = \|j(t)\|_1 \] leads to sparsity

- Technically: \( L_1 \)-norm leads to sparsity

Solution nonlinear in data, iterative optimization required

[Matsuura et al., 1995; Gorodnitsky et al., 1995]
Limitations of smooth and sparse inverses

Smooth inverses
- Difficulty to distinguish sources
- Occurrence of „ghost sources“
Limitations of smooth and sparse inverses

**Smooth inverses**
- Difficulty to distinguish sources
- Occurrence of “ghost sources”

**Sparse inverses**
- Scattered sources in the presence of noise
Combining sparsity and smoothness

1. Mixed-norm penalties, e.g.,

\[ g(j) = \| j(t) \|_1 + \gamma \| j(t) \|_2 \]

[Haufe et al., 2008; Vega-Hernández et al., 2008]
Combining sparsity and smoothness

1. **Mixed-norm penalties**, e.g.,
   \[ g(j) = \| j(t) \|_1 + \gamma \| \Gamma j(t) \|_2 \]
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2. **Sparsity in different spatial basis**

   E.g. \[ g(j(t)) = \| \tilde{j}_s(t) \|_1 \text{ with } j = \Phi_s \tilde{j}_s \text{ and } \Phi_s = \]

   \[ \rightarrow \text{Solution has simple spatial structure} \]
   [Haufe et al., 2011]
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   ![](image1)

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   [Haufe et al., 2011]
Comparison

Smooth

NOISELESS

NOISY

Both

Sparse
Localization of hand areas in somatosensory cortex

Electrical stimulation at both thumbs
(Median nerves)

→ N20 event-related potential in the EEG

There should be two lateralized symmetric sources in the somatosensory cortices.

[Haufe et al., 2008]
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Technicalities

- Compensating for bias towards superficial sources
- Fixing current orientations in cortically-constrained estimation
- Measuring distances on the cortical manifold
- Achieving sparsity for vectorial currents
- Dealing with time series data
Reconstruction of time series

Problem with L₁-norm penalties: sparsity pattern may differ for each sample, causing jumps in the source time series between voxels.

Figure from Gramfort et al., 2013
Reconstruction of time series

Problem with $L_1$-norm penalties: sparsity pattern may differ for each sample, causing jumps in the source time series between voxels.

Remedy for stationary time series: select the same set of active voxels/basis functions for all samples.

$$g \left( \tilde{j}(1), \ldots, \tilde{j}(T) \right) = \sum_{i} \left\| \left( \tilde{j}_i^T(1), \ldots, \tilde{j}_i^T(T) \right)^T \right\|_2 = \left\| \tilde{J} \right\|_{21}$$

[Haufe et al., 2008; Ou et al., 2009]

Figure from Gramfort et al., 2013
Reconstruction of time series

To model nonstationarity:

- Decompose time series into time-frequency atoms

\[ j = \tilde{j}_t \Phi_t \]

\[ \Phi_t = \]

- Mixed-norm penalty

\[ g(J) = \| \tilde{J}_t \|_2 + \gamma \| \tilde{J}_t \|_1 \]

[Gramfort et al., 2013]
Reconstruction of time series

To model nonstationarity:
• Decompose time series into time-frequency atoms

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\[ \Phi_t = \begin{array}{c}
\text{waveform} \\
\text{signal}
\end{array} \]

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Other dynamical constraints: Random walk model, Kalman filter, …

[Schmitt et al., 2002; Galka et al., 2004]

Figure from Gramfort et al., 2013
Other source localization paradigms

**Dipole fits:** instead of estimating currents of $N \gg M$ dipoles with fixed locations, estimate current+location of $K \ll M$ dipoles.

Nonconvex cost function, danger of local minima.  

[e.g., Scherg, 1992]
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**Scanning Techniques:**

- **Subspace methods** (MUSIC, RapMUSIC): for each voxel, compute angle between space spanned by dipole at that voxel and space spanned by data. The angle is taken as an index of activation at that voxel.

[Schmitt, 1986; Mosher and Leahy, 1999]
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  [Schmitt, 1986; Mosher and Leahy, 1999]

- **Beamformers:** for each voxel, find a spatial filter which maximizes the SNR of signals originating at that voxel. The SNR at each voxel is taken as an activity index.

  [van Veen et al., 1997]

Activity indices of scanning techniques do not explain the data.
If temporal constraints are available, one might drop spatial constraints.

→ Useful if no accurate leadfield (e.g., no individual structural MRI) exists.
(Blind) source separation

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Factorize current density into \( j(t) = Fs(t) + \epsilon_j(t) \), where

\[ s(t) \in \mathbb{R}^K \]

are \( K \leq M \ll N \) latent factors (sources, components) of brain activity with specific temporal dynamics, and

\[ F \in \mathbb{R}^{3N \times K} \]

are their corresponding source space activation patterns.
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The overall decomposition of the EEG becomes

\[
x(t) = LF_s(t) + L\epsilon_j(t) + \epsilon(t) = As(t) + \epsilon(t)
\]

where

\[ A \in \mathbb{R}^{M \times K} \]

are sensor-space activation patterns to be estimated.
The factor/component time series

\[ x(t) = A_s(t) + \epsilon(t) \]
The factor/component time series

\[ x(t) = As(t) + \varepsilon(t) \]

\( s(t) \in \mathbb{R}^K \), to be estimated

Each \( s_k(t) \) is linked to a static sensor-space activation pattern \( a_k \) rather than to a brain location.
The sensor space activation patterns

\[ x(t) = As(t) + \varepsilon(t) \]

\[ A \in \mathbb{R}^{M \times K}, \text{ also to be estimated} \]

Columns:

\[ a_1 \quad a_2 \quad a_3 \]

The activation patterns \( a_k \) represent the time invariant current density of the component \( s_k(t) \).
Source localization of activation patterns

\[ x(t) = As(t) + \varepsilon(t) = LF s(t) + \varepsilon(t) \]

Recall that \( a_k = L f_k \).  

\( \rightarrow \) Using the techniques described in the first part, the estimated \( a_k \) can be source-localized by estimating \( f_k \) using a precomputed leadfield \( L \).  

The source space activation patterns \( f_k \) represent the time invariant current density of the component \( s_k(t) \).
Forward and backward models

A BSS method may either directly fit the **forward model** $x(t) = As(t) + \varepsilon(t)$

(that is, estimate $A$ and $s(t)$ jointly),
Forward and backward models

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or fit a **backward model** \( W^\top x(t) = s(t) \)

parameterized only by the **extraction filters** \( W \in \mathbb{R}^{M \times K} \).
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If \( \varepsilon(t) \) and \( s(t) \) are uncorrelated, both approaches are equivalent,

and related through \( A = \Sigma_x W \Sigma_s^{-1} \),

where \( \Sigma_x \) and \( \Sigma_s \) are the covariance matrices of \( x(t) \) and \( s(t) \). [Parra et al., 2005; Haufe et al., 2014]
Forward and backward models

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parameterized only by the **extraction filters** $\mathbf{W} \in \mathbb{R}^{M \times K}$.

If $\varepsilon(t)$ and $\mathbf{s}(t)$ are uncorrelated, both approaches are equivalent,

and related through $\mathbf{A} = \Sigma_x \mathbf{W} \Sigma_s^{-1}$, \[^{[\text{Parra et al., 2005; Haufe et al., 2014}]}\]

where $\Sigma_x$ and $\Sigma_s$ are the covariance matrices of $\mathbf{x}(t)$ and $\mathbf{s}(t)$.

Both forward and backward models provide solutions of the inverse problem, as long as $\mathbf{x}(t)$ is "raw" (not nonlinearly preprocessed) EEG data.
Parameter interpretation

Both filters and patterns can be visualized as scalp maps. However, their meanings are completely different.

[Haufe et al., 2014]
Parameter interpretation

Both filters and patterns can be visualized as scalp maps. However, their meanings are completely different.

Patterns tell us how brain activity $s_k(t)$ is expressed in each sensor.

$\rightarrow$ $a_k$ depends only on $s_k(t)$. 

[Haufe et al., 2014]
**Parameter interpretation**

Both filters and patterns can be visualized as scalp maps. However, their meanings are completely different.

- **Patterns** tell us how brain activity $s_k(t)$ is expressed in each sensor. \[ a_k \]  
  \[ \rightarrow a_k \text{ depends only on } s_k(t). \]

- **Filters** tell us how to weight sensors to extract the brain activity $s_k(t)$. \[ w_k \]  
  \[ \rightarrow w_k \text{ depends on } s_k(t) \text{ and all noise sources.} \]

[Haufe et al., 2014]
Parameter interpretation

Both filters and patterns can be visualized as scalp maps. However, their meanings are completely different.

Patterns tell us how brain activity $s_k(t)$ is expressed in each sensor.

$\rightarrow a_k$ depends only on $s_k(t)$.

Only patterns can be source localized by virtue of $a_k = L f_k$.

Filters tell us how to weight sensors to extract the brain activity $s_k(t)$.

$\rightarrow w_k$ depends on $s_k(t)$ and all noise sources.

[Haufe et al., 2014]
BSS methods

For model fitting, a backward modeling approach is typically adopted,

\[
W = \arg \min_{W'} f \left( W'^{\top} x(t) \right)
\]

where \( f \) encodes assumptions on the sources \( s(t) = W^{\top} x(t) \).
BSS methods

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\[ \mathbf{W} = \arg \min_{\mathbf{W}'} f \left( \mathbf{W}'^T \mathbf{x}(t) \right) \]

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<tr>
<td>CCA</td>
<td>SCSA</td>
</tr>
<tr>
<td>CSP</td>
<td>MVARICA</td>
</tr>
<tr>
<td>SPoC</td>
<td>CICAAR</td>
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<tr>
<td>cSPoC</td>
<td>PISA</td>
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<tr>
<td>SSD</td>
<td>MOCA</td>
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<td>DSS</td>
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</table>
BSS methods by assumption

Brain activity differs between experimental conditions. (ERP studies)
→ Linear classifiers (LDA, SVM, LLR)

[e.g., Blankertz et al., 2010]
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Brain activity correlates with behaviour or stimulus variables.
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Brain activity of interest correlates across subjects/stimulus repetitions.
(Hyperscanning ERP studies)
→ Canonical correlation analysis (CCA)  
[e.g., Dmochowski et al., 2011]
BSS methods by assumption (2)

Brain components are mutually independent.
   (many uses including artifact removal)
→ Independent component analysis (ICA)

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If the brain activity of interest can be characterized in several ways, multiple BSS methods may lead to the same solution.

---

[e.g., Makeig et al.]

[Gomez-Herrero et al., 2008; Haufe et al., 2010]

[von Bünau et al., 2009]
Not all EEG phenomena are phase-locked to certain events. There are also **rhythms**, the amplitude of which modulates depending on the mental state.

Most rhythms are **idle** rhythms, i.e., are attenuated during activation.

- **α**-rhythm (around 10 Hz) in visual cortex:

  - eyes closed:

  - eyes open:

- **μ**-rhythm (around 10 Hz) in motor and sensory cortex:

  - arm at rest:

  - arm moves:

Figures by Benjamin Blankertz
Spatio-spectral decomposition (SSD)

Signal of interest is narrow-band oscillation.

\[
\mathbf{w} = \arg \max_{\mathbf{w}'} \text{SNR}(\mathbf{w}')
\]

\[
= \arg \max_{\mathbf{w}'} \frac{\mathbf{w}'^\top \Sigma_{\text{signal}} \mathbf{w}'}{\mathbf{w}'^\top (\Sigma_{\text{noise}}) \mathbf{w}'}
\]

\(\Sigma_{\text{signal}}\) and \(\Sigma_{\text{noise}}\) are the covariances of the data filtered in the central and flanking frequency bands.

\(\mathbf{w}\) is obtained as the solution to the generalized eigenvalue equation

\[
\Sigma_{\text{signal}} \mathbf{w} = \lambda \Sigma_{\text{noise}} \mathbf{w} \quad (\text{in Matlab: } \mathbf{W} = \text{eig}(\Sigma_{\text{signal}}, \Sigma_{\text{noise}});)
\]

[Nikulin et al., 2011]
Common spatial patterns (CSP)

Power of oscillations differs between two experimental conditions C1 and C2.

\[ w_1 = \arg \min_w \frac{w^\top \Sigma_1 w}{w^\top (\Sigma_1 + \Sigma_2) w} \quad w_2 = \arg \min_w \frac{w^\top \Sigma_2 w}{w^\top (\Sigma_1 + \Sigma_2) w} \]

[Koles et al., 1990]
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**Example:** BCI based on motor imagery of left and right hand.

Figures by Benjamin Blankertz
Source power correlation analysis (SPoC)

Instantaneous amplitude (envelope) of oscillations correlates with continuous variable (behaviour, stimulus properties, etc.) .

\[ w = \arg \max_{w'} \text{corr} \left( \text{env} \left( w'^T x(t) \right), z(t) \right) \]

[Dähne et al., 2014]
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Instantaneous amplitude correlates across subjects/stimulus repetitions
\[ \rightarrow \text{Canonical SPoC (cSPoC)} . \]

[Dähne et al., 2014, Submitted]
Extraction of steady-state auditory evoked potentials

Rhythmic auditory stimulation elicits phase-locked rhythmic activity in auditory cortex = SSAEP (same as for visual stimulation and SSVEP).

[e.g., Galambos et al., 1981]

Linear relationship between loudness (in dB) and SSAEP amplitude.

[Picton et al., 2003]
Extraction of steady-state auditory evoked potentials

Rhythmic auditory stimulation elicits phase-locked rhythmic activity in auditory cortex = SSAEP (same as for visual stimulation and SSVEP).

Linear relationship between loudness (in dB) and SSAEP amplitude.

**Experiment:** apply 40Hz artificial sound stimulus modulating loudness.

**Task:** identify SSAEP component.
Extraction of steady-state auditory evoked potentials

Results:
• Compared to single sensors, SPoC leads to higher SNR (peak height) and higher correlation with the sound volume ($r=0.6$ vs. $r=0.1$)
• SPoC activation pattern localizes to left and right auditory cortices
• Similar results for SSD instead of SPoC

[Dähne et al., 2014]
Summary

- EEG data are mixed due to volume conduction in the head

- To increase SNR, and achieve interpretability, the inverse problem needs to be "solved"

- Can be done using a physical model of volume conduction (inverse source reconstruction) or using purely statistical models (source separation)

- In any case, a unique solution is only obtained if prior assumptions/constraints are imposed

- Correctness of the solution relies on correctness of assumptions
Origin of blurring

- Both sources explain data equally well
- Source 1 has $L_2$-norm: $\sqrt{1^2 + 1^2} = \sqrt{2}$
- Source 2 has $L_2$-norm: $\sqrt{2^2} = 2$
The level sets of Likelihood and constraint \textit{almost always} intersect at the coordinate axes.
No sparsity using $L_2$-norm

The level sets of Likelihood and constraint *almost never* intersect at the coordinate axes.
Depth compensation

Superficial sources contribute more to the EEG than deep ones.

→ Many superficial sources „cost less“ than one deep source.

→ Location bias towards superficial sources.
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**Countermeasure:** minimize norm of weighted sources

\[ g(s) = \| Ws \| \]

with diagonal or blockdiagonal \( W \) encoding a voxel-specific penalty
Depth compensation

1. Norm of the columns of the lead field

2. Voxel-wise (co-) variance of the minimum-norm solution

3. Norm + distance from EEG sensors

Choice of $W$ is crucial.

[Jeffs et al., 1987]

[Pascual-Marqui, 2002; Haufe et al., 2008]

[Marzetti et al., 2008]
Sparsity of Vector Fields

Dipole orientations are 3D vectors, current distributions are 3D vectorfields

**Technicallity:** $L_1$-norm sets single dimensions to 0

→ Estimated sources are not physiologically plausible (parallel to coordinate axes)

[Haufe et al., 2008; Ding et al., 2008; Ou et al., 2009]
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More „physiological“ constraints

1. Sources on cortex, arbitrary orientation
2. Sources on cortex, orientation normal to surface (dangerous!)
3. Regions of interest
4. Symmetric configurations