How to Hedge an Option Against an Adversary: Black-Scholes Pricing is Minimax Optimal

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NIPS 2013 Spotlight
Black-Scholes Option Pricing

- Given a financial contract with known payoff at time $T$, how much is it worth now?
- **Black-Scholes (1973):** model asset price as continuous-time random walk in log space
- Dynamic hedging strategy $\rightarrow$ replicate option payoff $\rightarrow$ fair price

$$\text{price at time } 0 = \mathbb{E}[\text{payoff (asset price at time } T)]$$

Geometric Brownian Motion
Robust Option Pricing via Regret Minimization

- Hedging strategy \( \equiv \) online learning algorithm
- Can we construct a trading strategy that is robust to adversarially chosen price?

Investor
- Observes asset price \( S \)
- Invests \( \Delta \)

Market
- Selects fluctuation \( r \)
- Updates price \( S \leftarrow S(1+r) \)

Investor profits \( \Delta r \)

- Investor’s goal is to minimize his regret:

\[
g(S \cdot \prod_{i=1}^{n} (1 + r_i)) - \sum_{i=1}^{n} \Delta_i r_i
\]

- payoff of option
- profit from trading

- Optimal regret is equivalent to “minimax value of option”
Black-Scholes Price is Minimax Optimal

- Analyze minimax regret:

\[ V^n_\zeta(S, c) = \inf_{\Delta_1} \sup_{r_1} \ldots \inf_{\Delta_n} \sup_{r_n} \left\{ g(S \cdot \prod_{i=1}^{n} (1 + r_i)) - \sum_{i=1}^{n} \Delta_i r_i \right\} \]

with **cumulative volatility** and **maximum jump** constraints:

\[ \sum_{i=1}^{n} r_i^2 \leq c, \quad |r_i| \leq \zeta \]

**Theorem:** If payoff \( g \) is convex and Lipschitz, and \( \zeta_n \to 0 \), then

\[ \lim_{n \to \infty} V^n_\zeta(S, c) = U(S, c) \quad [\text{Black-Scholes price}] \]

- Explicit trading strategy gives finite-horizon upper bound on regret:

\[ V^n_\zeta(S; c) \leq U(S, c) + O(c \zeta^{1/4}) \]