Submodular Function Minimization

Satoru Iwata
(RIMS, Kyoto University)

Outline

- Submodular Functions
  - Examples
  - Discrete Convexity
- Submodular Function Minimization
  - Min-Max Theorem
  - Combinatorial Algorithms
- Applications
- Approximation Algorithms
- Conclusion

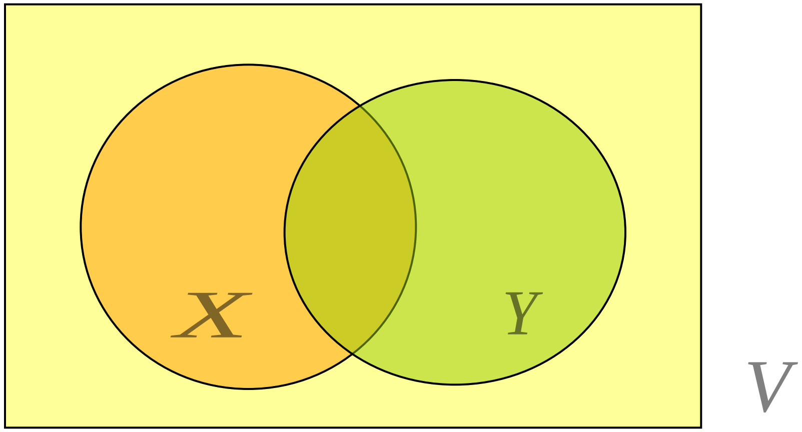
Submodular Functions

V : Finite Set

$f : 2^V \to \mathbb{R}$ $\qquad \forall X, Y \subseteq V$

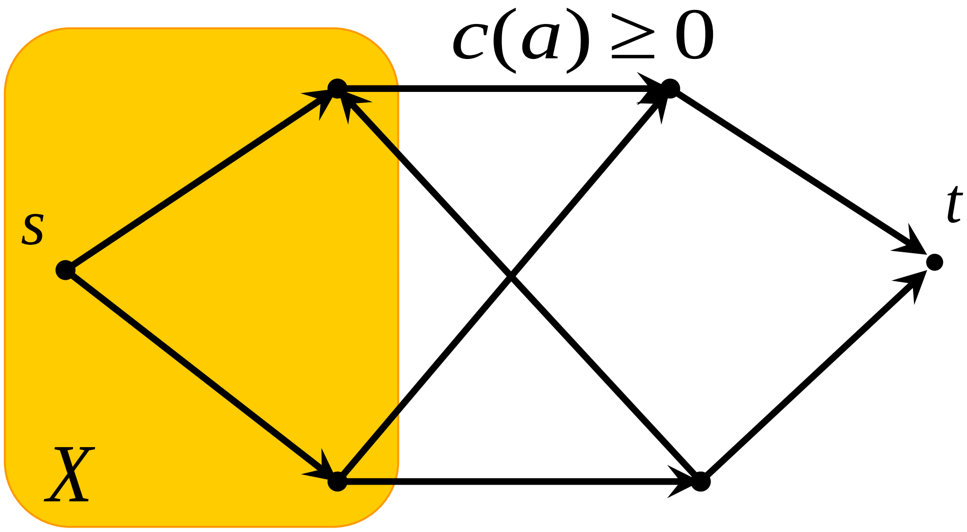
$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

- Cut Capacity Functions
- Matroid Rank Functions
- Entropy Functions



Cut Capacity Function

Cut Capacity $\kappa(X) = \sum \{c(a) \mid a : \text{leaving } X\}$



Max Flow Value＝Min Cut Capacity

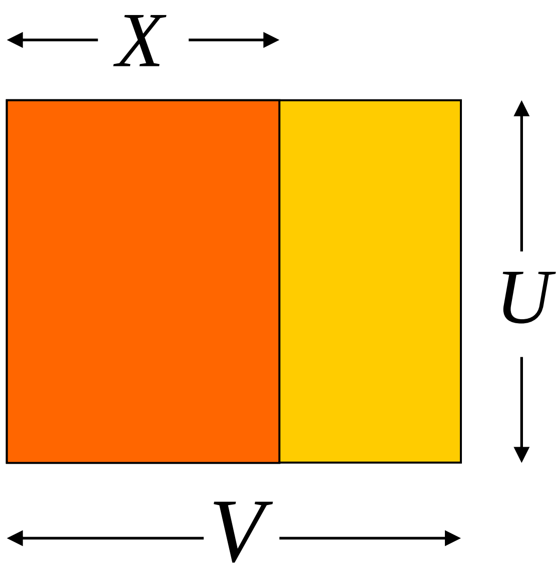
Matroid Rank Functions

Matrix Rank Function

Whitney (1935)

$$\rho(X) = \operatorname{rank} A[U, X]$$

$A =$



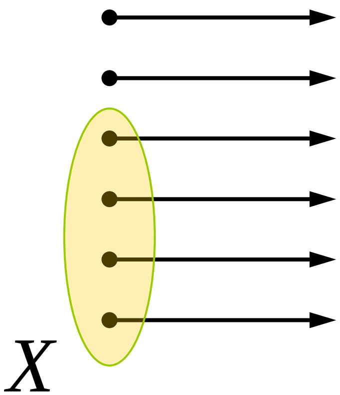
$$\forall X \subseteq V, \rho(X) \leq |X|$$

$$X \subseteq Y \Rightarrow \rho(X) \leq \rho(Y)$$

ρ : Submodular

Entropy Functions

Information Sources



$h(\phi) = 0$

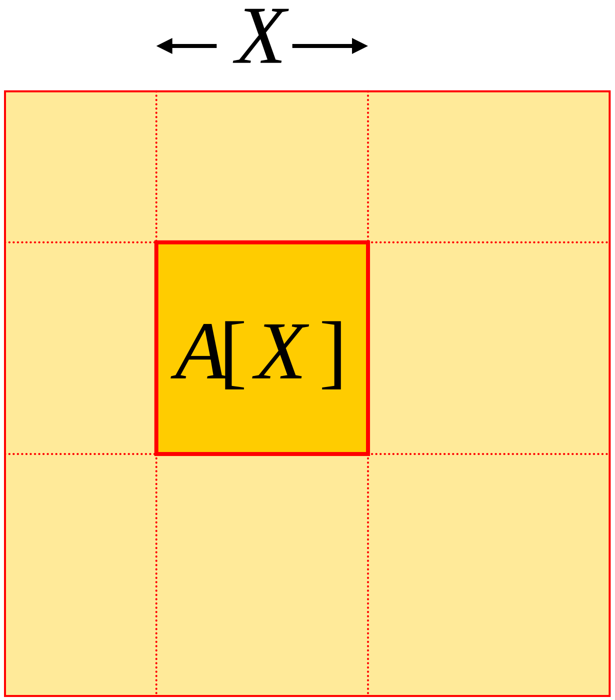
$h(X)$: Entropy of the Joint Distribution

$h(X) + h(Y) \geq h(X \cap Y) + h(X \cup Y)$

Conditional Mutual Information ≥ 0

Positive Definite Symmetric Matrices

$A =$



$$f(\phi) = 0$$

$$f(X) = \log \det A[X]$$

Ky Fan's Inequality

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

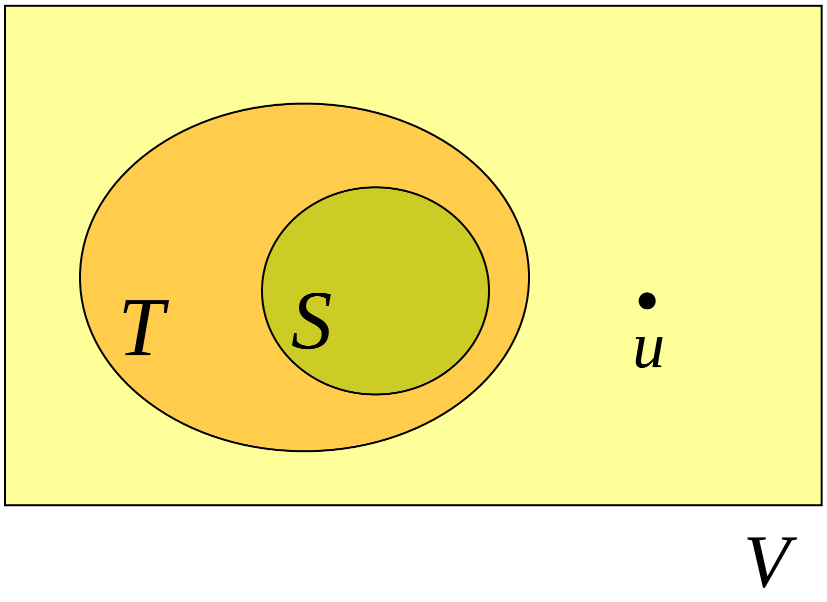
Extension of the Hadamard Inequality

$$\det A \leq \prod_{i \in V} A_{ii}$$

Discrete Concavity

$$S \subseteq T \Rightarrow$$

$$f(S \cup \{u\}) - f(S) \geq f(T \cup \{u\}) - f(T)$$

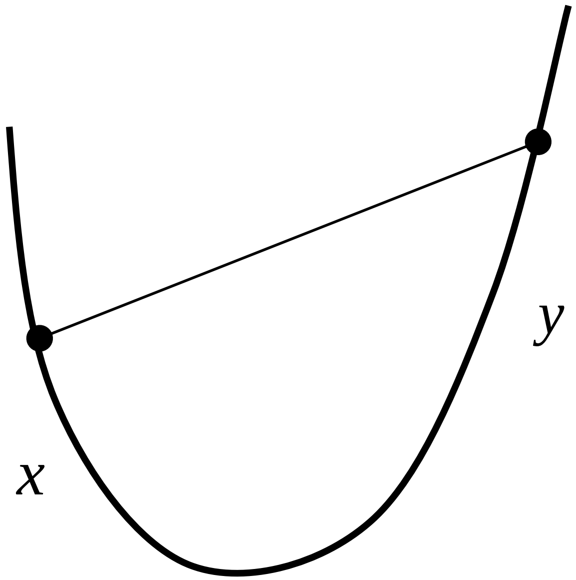


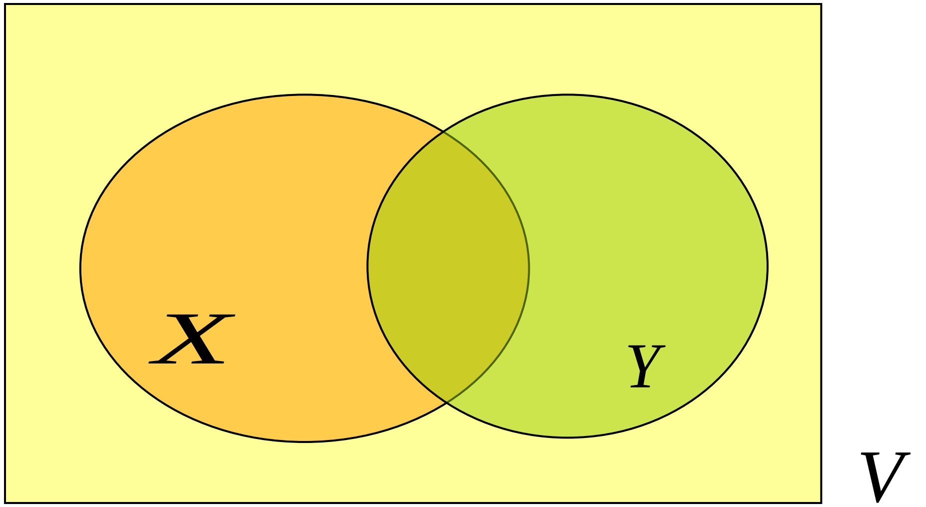
Diminishing Returns

Discrete Convexity

Convex Function

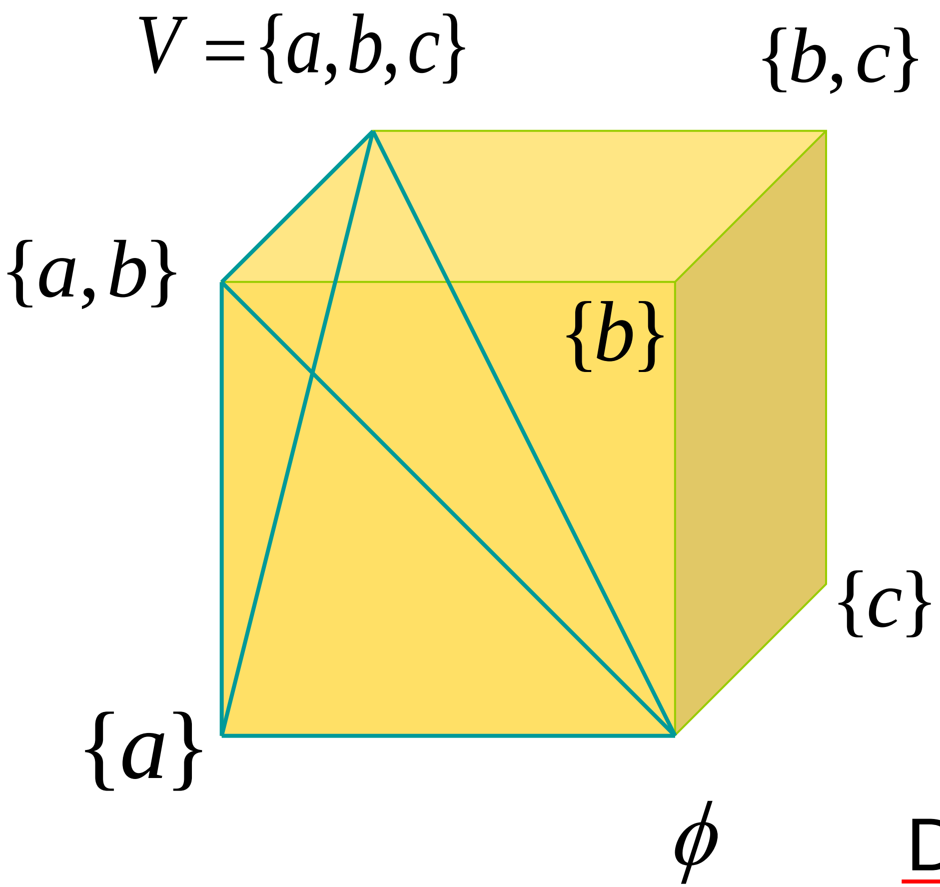
$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$





Discrete Convexity

Lovász (1983)



$\hat{f}$: Linear Interpolation

$\hat{f}$: Convex

⇕

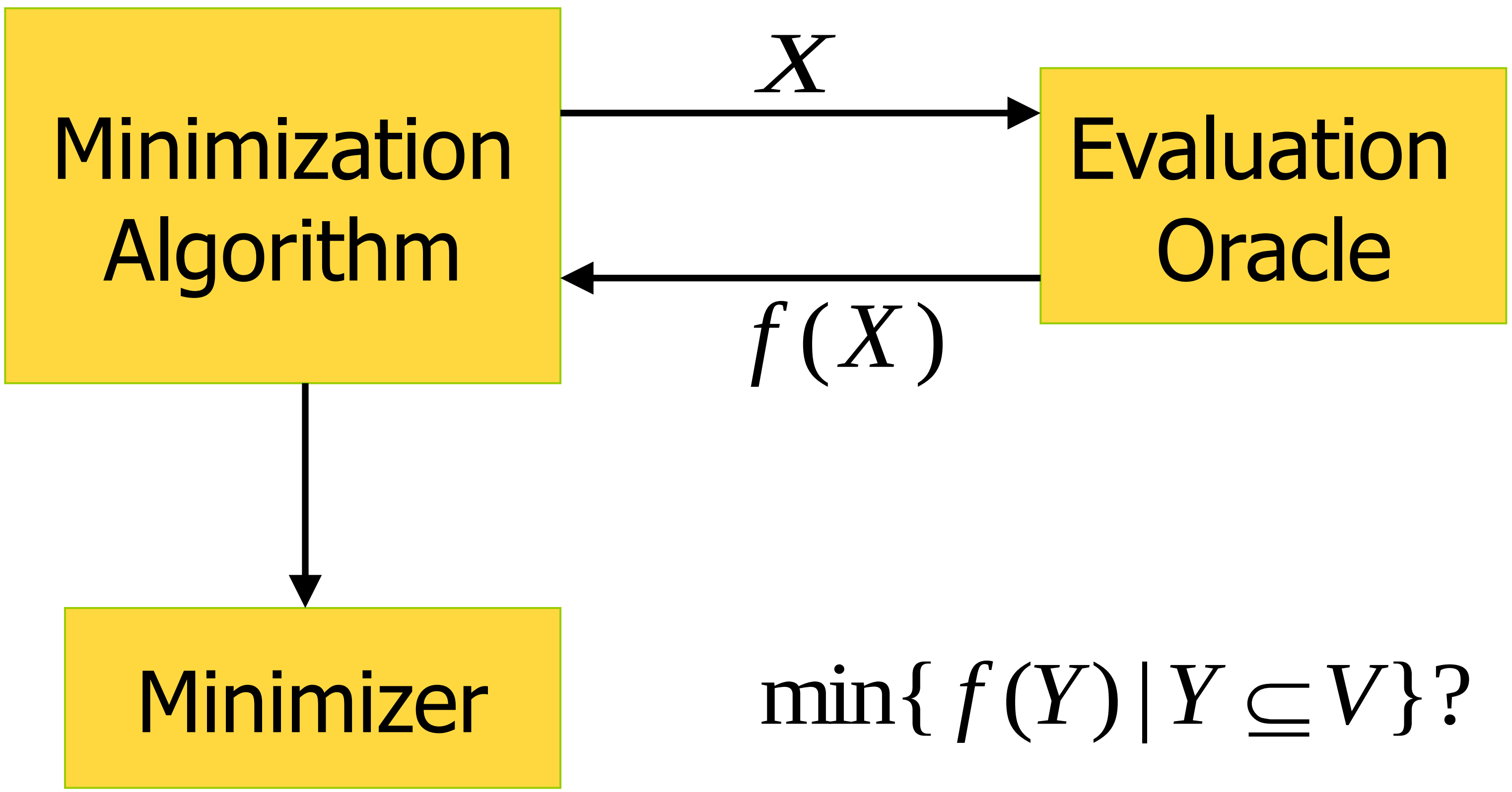
f : Submodular

Discrete Convex Analysis

Murota (2003)

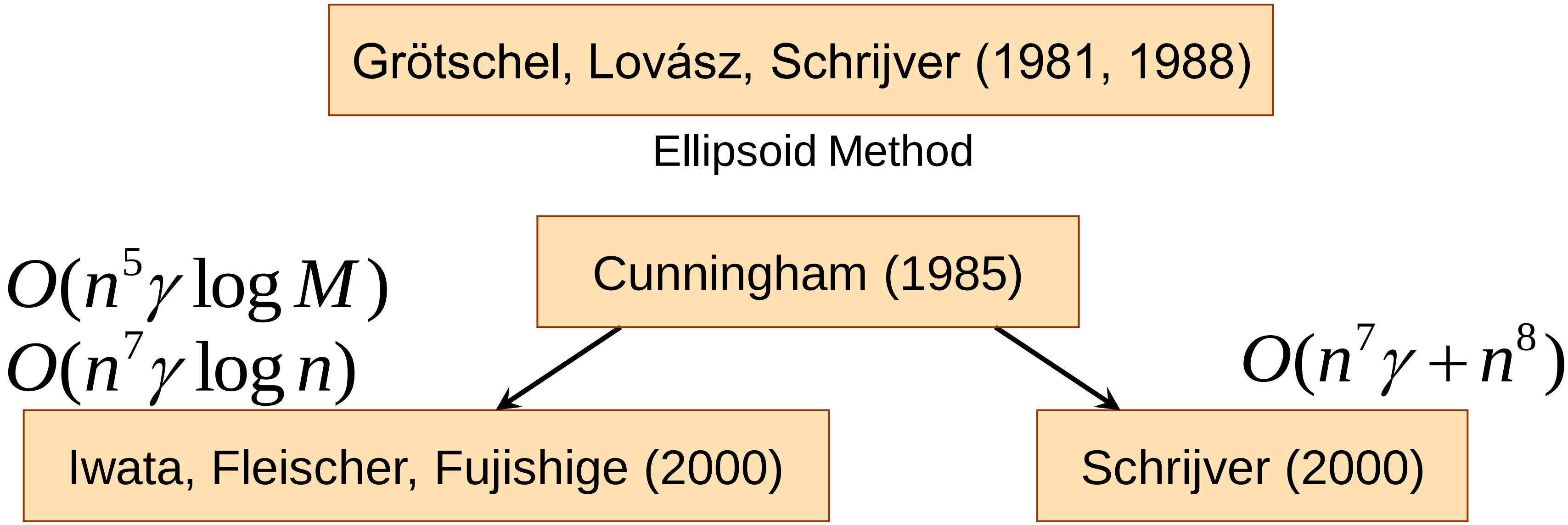
Submodular Function Minimization

Assumption: $f(\phi) = 0$



$$\min\{ f(Y) \mid Y \subseteq V \}?$$

Submodular Function Minimization

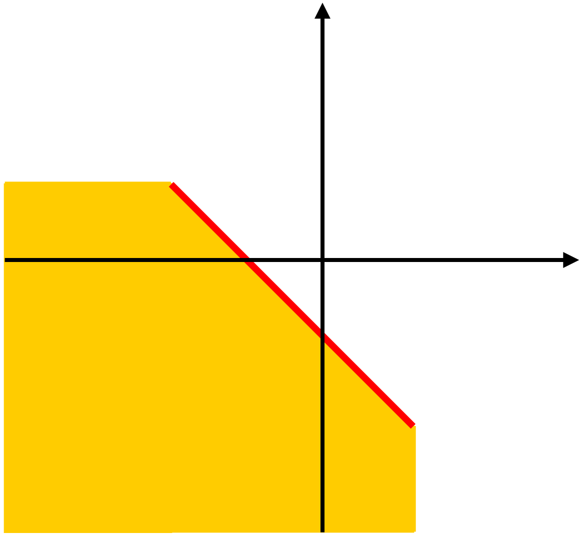


γ : Time for Function Evaluation

$$M = \max_{X \subseteq V} |\, f(X) \,|$$

Base Polyhedra

$$\mathbf{R}^V = \{x \mid V \to \mathbf{R}\}$$



$$x(Y) = \sum_{v \in Y} x(v)$$

Submodular Polyhedron

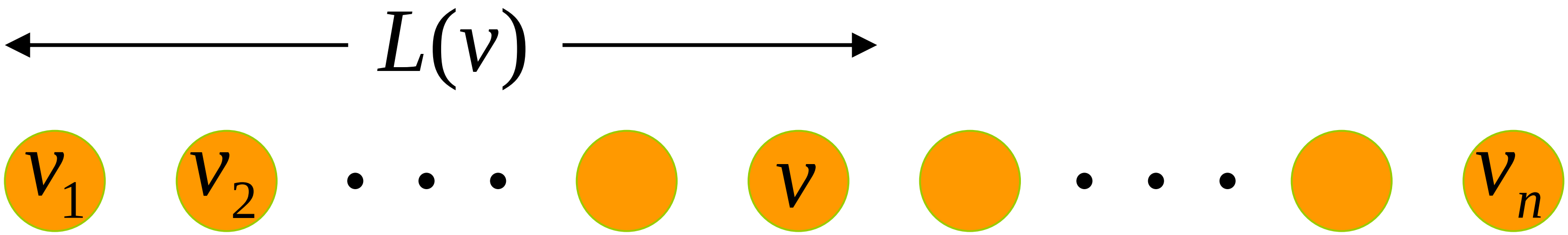
$$P(f) = \{x \mid x \in \mathbf{R}^V, \forall Y \subseteq V, x(Y) \le f(Y)\}$$

Base Polyhedron

$$B(f) = \{x \mid x \in P(f), x(V) = f(V)\}$$

Greedy Algorithm

Edmonds (1970)
Shapley (1971)



$$y(v) = f(L(v)) - f(L(v) - \{v\}) \qquad (v \in V)$$

y : Extreme Base

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y(v_1) \\ y(v_2) \\ \vdots \\ y(v_n) \end{bmatrix} = \begin{bmatrix} f(L(v_1)) \\ f(L(v_2)) \\ \vdots \\ f(L(v_n)) \end{bmatrix}$$

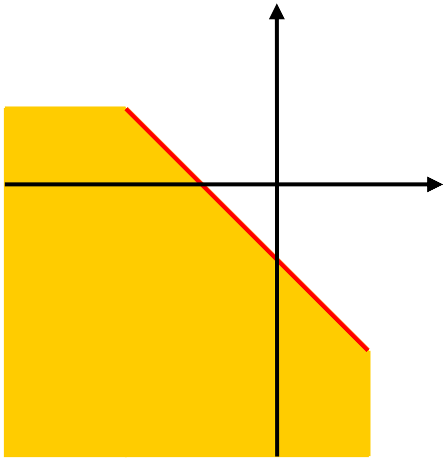
Min-Max Theorem

Theorem Edmonds (1970)

$$\min_{Y \subseteq V} f(Y) = \max\{ x^-(V) \mid x \in B(f) \}$$

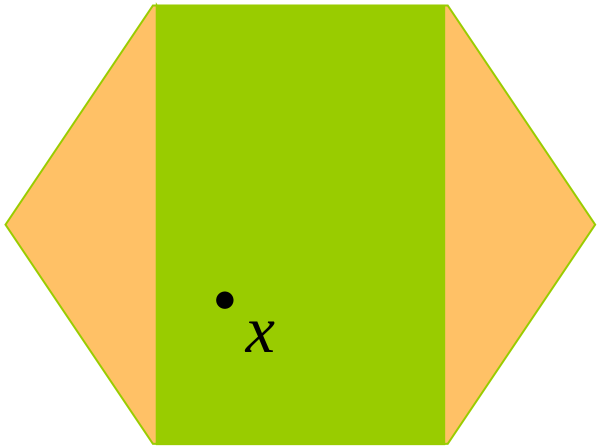
$$x^-(v) := \min\{0, x(v)\}$$

$$x^-(V) \le x(Y) \le f(Y)$$



Combinatorial Approach

Extreme Base $y_L \in B(f)$



Convex Combination

$$x = \sum_{L \in \Lambda} \lambda_L y_L$$

Cunningham (1985)

$$O(n^6 M\gamma \log nM)$$

$$M = \max_{X \subseteq V} | f(X) |$$

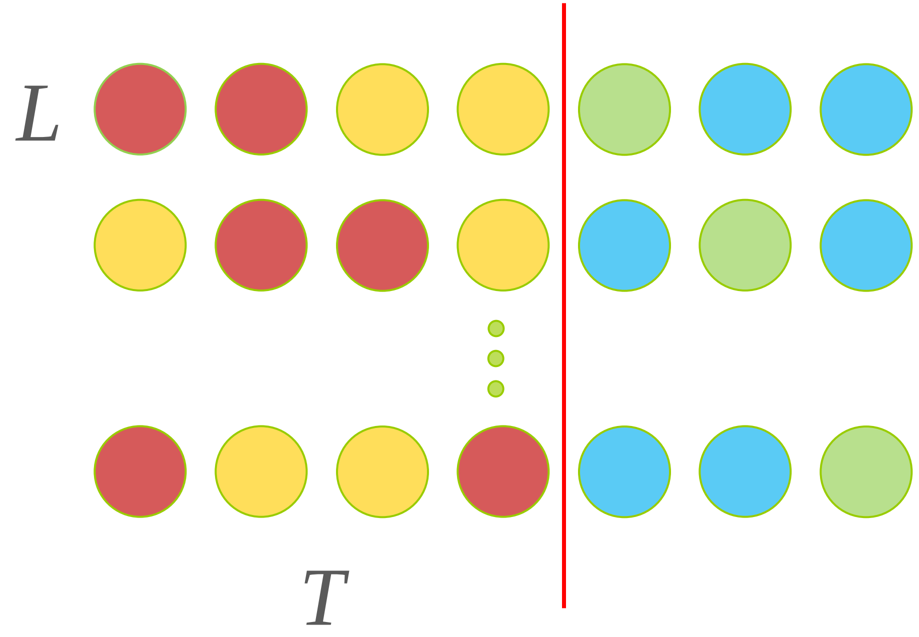
Combinatorial Approach

$$x = \sum_{L \in \Lambda} \lambda_L y_L$$

y_L : Extreme Base

$$x(v) \le 0, \quad \forall v \in T$$

$$x(v) \ge 0, \quad \forall v \notin T$$

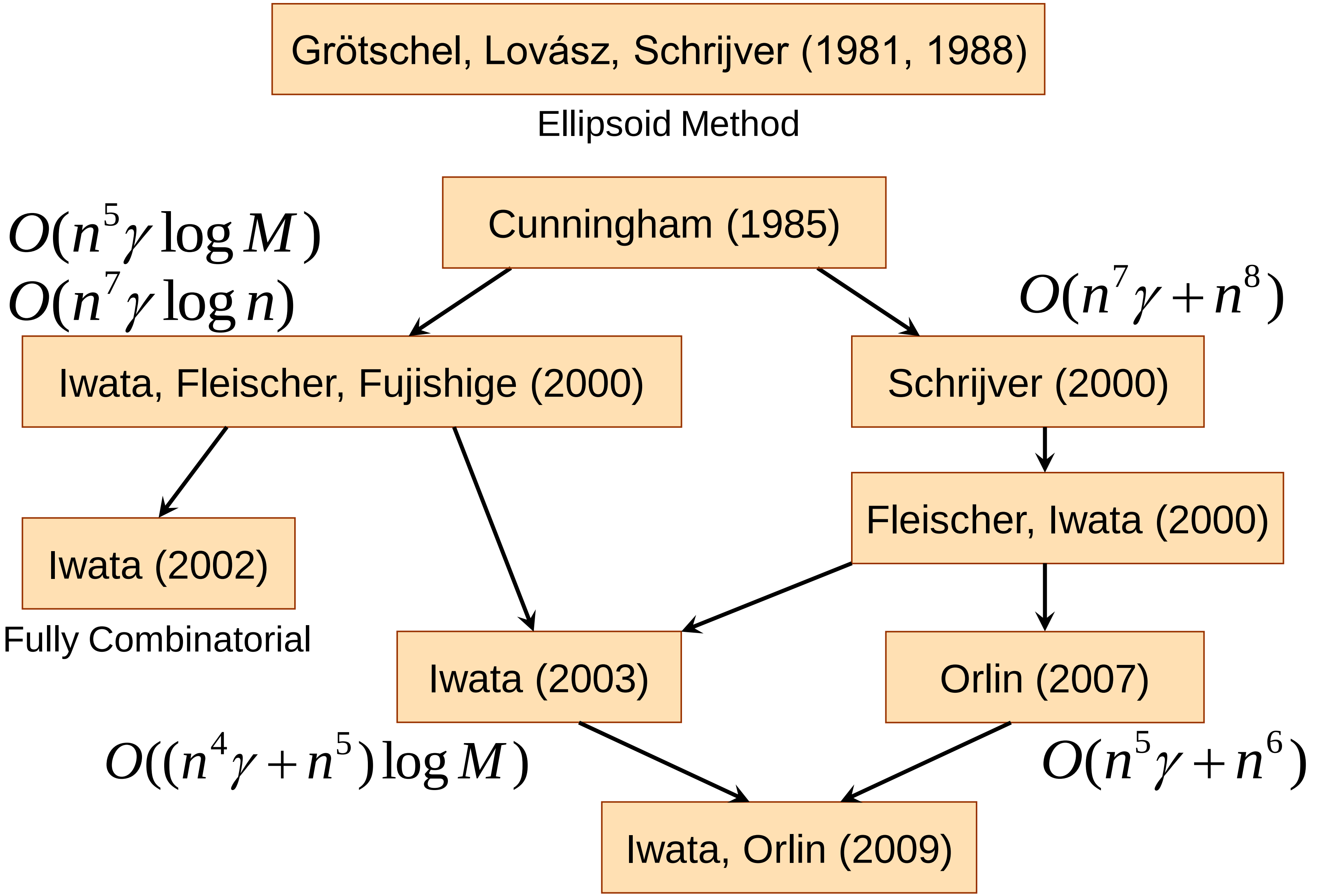


$$y_L(T) = f(T), \quad \forall L \in \Lambda. \quad \therefore x(T) = f(T).$$

$$x^-(V) = x(T) = f(T)$$

T : Minimizer

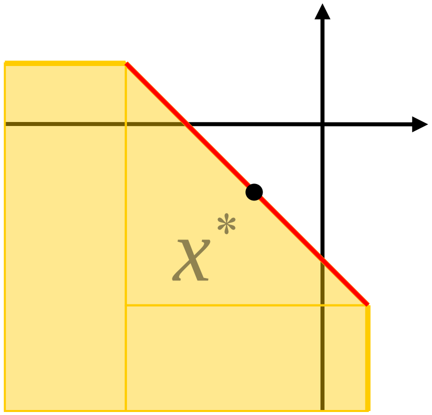
Submodular Function Minimization



The Fujishige-Wolfe Algorithm

$$\text{Minimize } \|x\|^2$$
$$\text{subject to } x \in B(f)$$

Fujishige (1984)



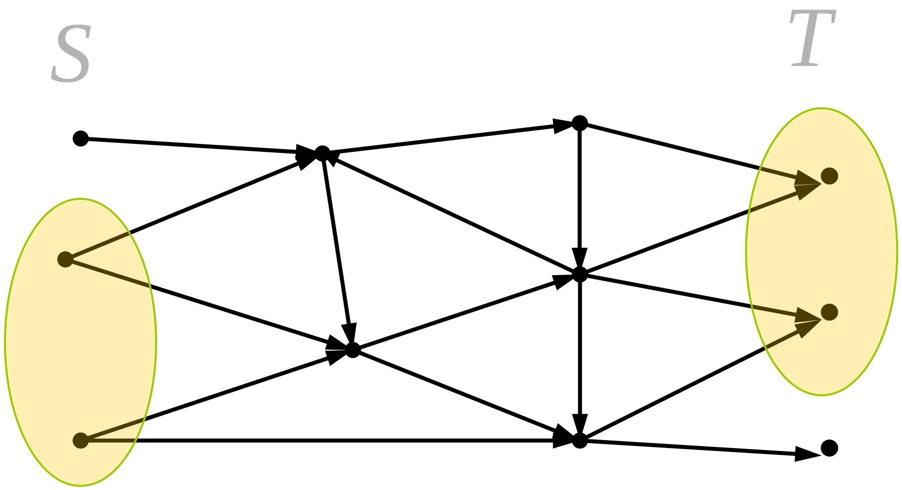
x^* : opt.sol.

$S := \{v \mid x^*(v) < 0\}$ ⟶ Minimal Minimizer

$T := \{v \mid x^*(v) \leq 0\}$ ⟶ Maximal Minimizer

$$f(S) = f(T) = \min_{X \subseteq V} f(X)$$

Evacuation Problem（Dynamic Flow）



Hoppe, Tardos (2000)

$c(a)$: Capacity

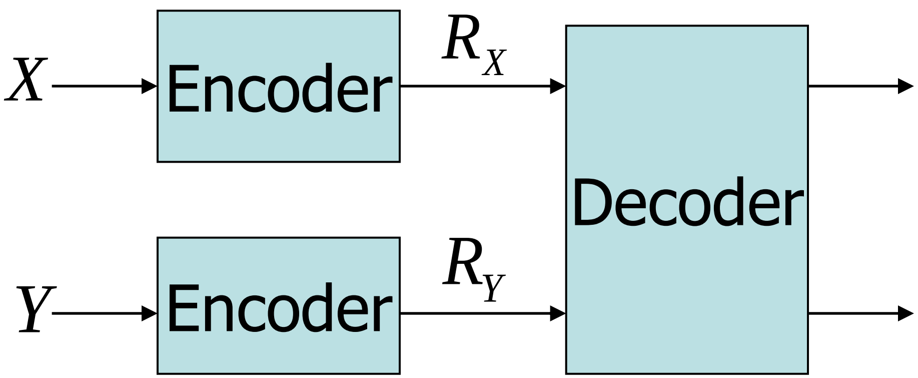
$\tau(a)$: Transit Time

$b(v)$: Supply/Demand

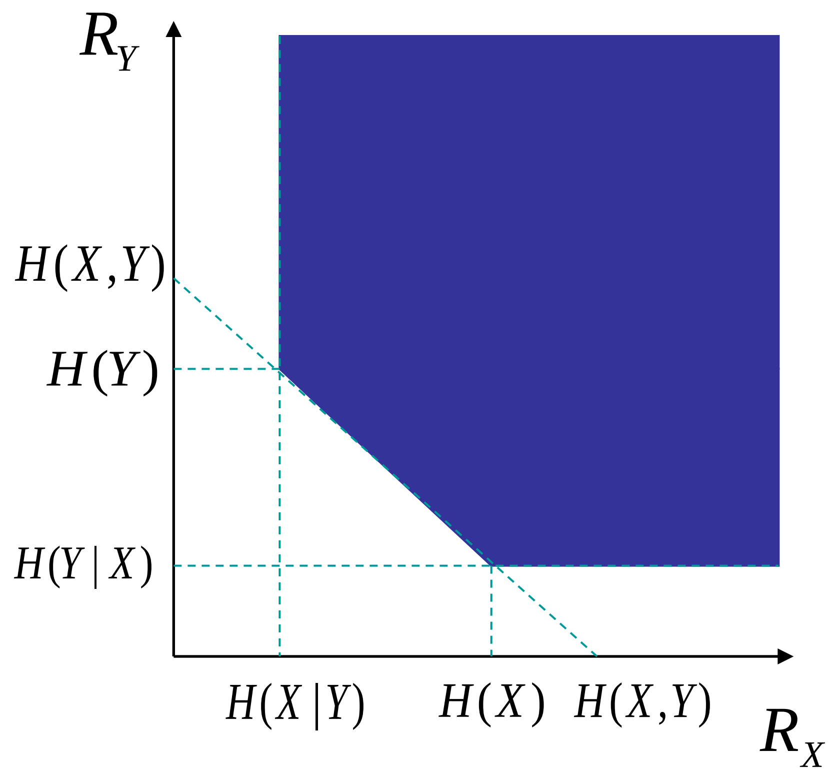
$o(X)$: Maximum Amount of Flow from $X \cap S$ to $T \setminus X$.

Feasible $\Longleftrightarrow$ $b(X) \leq o(X), \forall X \subseteq S \cup T$

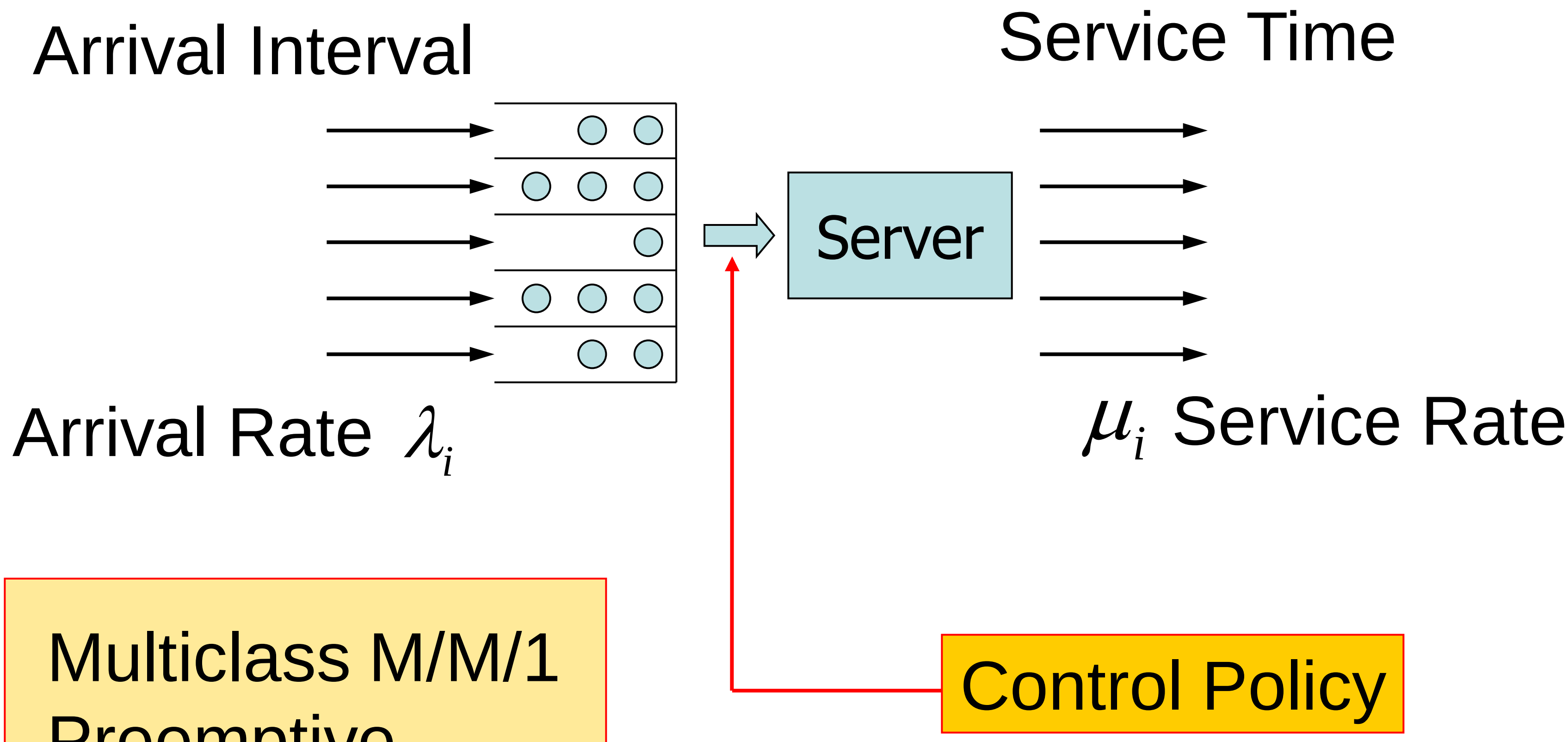
Multiterminal Source Coding



Slepian, Wolf (1973)



Multiclass Queueing Systems



Multiclass M/M/1
Preemptive

Performance Region

S_j : Expected Staying Time of a Job in *j*

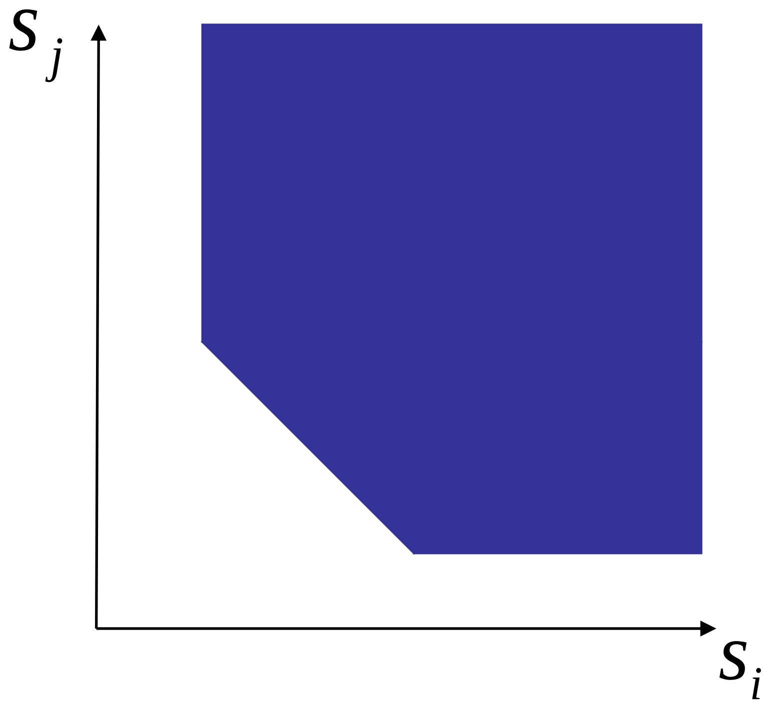
S : Achievable

⇕

$$\sum_{i \in X} \rho_i S_i \geq \frac{\sum_{i \in X} \rho_i / \mu_i}{1 - \sum_{i \in X} \rho_i}, \forall X \subseteq V$$

Coffman, Mitrani (1980)

$$\rho_i := \lambda_i / \mu_i, \quad \sum_{i \in V} \rho_i < 1$$



A Class of Submodular Functions

$x, y, z \in \mathbb{R}_+^V$

Itoko & Iwata (2007)

h : Nonnegative, Nondecreasing, Convex

$$f(X) = z(X) - y(X)h(x(X)) \qquad (X \subseteq V)$$

→ Submodular

$$\sum_{i \in X} \rho_i S_i \geq \frac{\sum_{i \in X} \rho_i / \mu_i}{1 - \sum_{i \in X} \rho_i}, \forall X \subseteq V$$

$$z_i := \rho_i S_i \qquad y_i := \frac{\rho_i}{\mu_i}$$

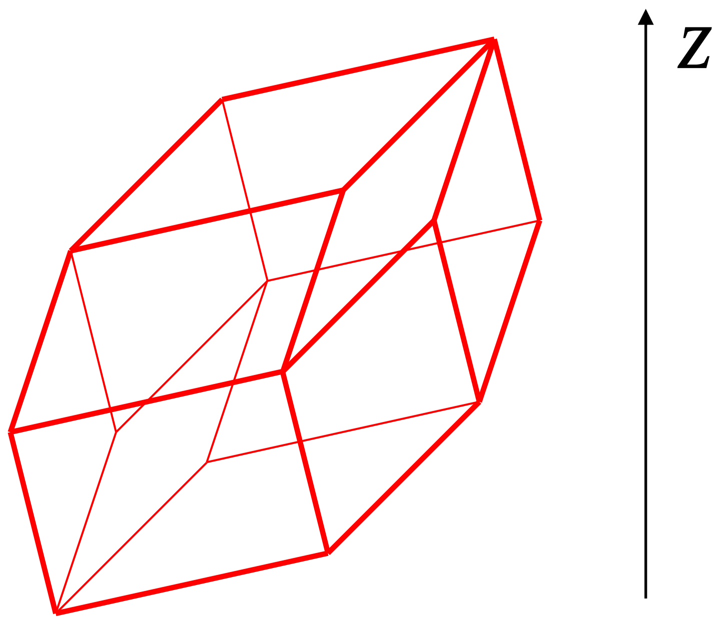
$$x_i := \rho_i \qquad h(x) := \frac{1}{1-x}$$

Zonotope in 3D

$$w(X) = (x(X), y(X), z(X))$$

$$Z = \mathrm{conv}\{w(X) \mid X \subseteq V\}$$

Zonotope

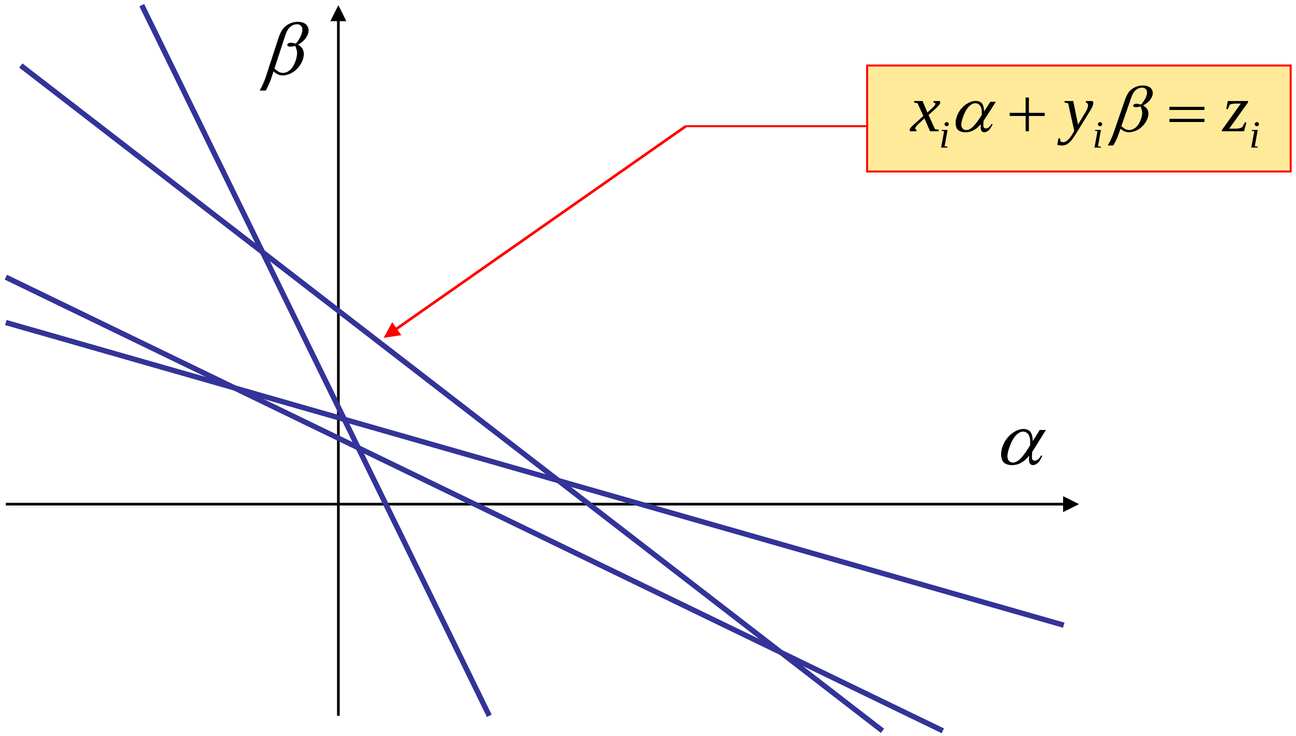


$$\tilde{f}(x, y, z) = z - yh(x)$$

$$\min\{ f(X) \mid X \subseteq V\}$$
$$= \min\{ \tilde{f}(x, y, z) \mid (x, y, z) : \text{Lower Extreme Point of } Z\}$$

Remark: $\tilde{f}(x, y, z)$ is NOT concave!

Line Arrangement



Enumerating All the Cells

Topological Sweeping Method
Edelsbrunner, Guibas (1989)

→ $O(n^2)$

Questions

What Kind of Approximation Algorithms
Can Be Extended to Optimization Problems
with Submodular Cost or Constraints ?

Cf. Submodular Flow (Edmonds & Giles, 1977)

Svitkina & Fleischer (FOCS 2008)

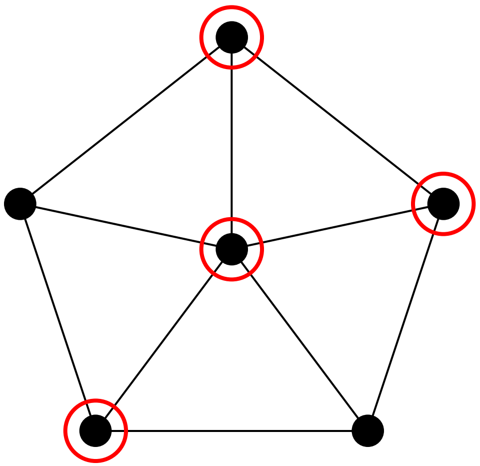
Randomized Sampling / Lower Bounds

Submodular Load Balancing

Submodular Sparsest Cut

Submodular Vertex Cover

Graph $G = (V, E)$



Submodular Function

$$f : 2^V \to \mathbf{R}_+$$

Find a Vertex Cover $S \subseteq V$ Minimizing $f(S)$

2-Approximation Algorithm

Goel, Karande, Tripathi, Wang (FOCS 2009)

Iwata & Nagano (FOCS 2009)

Relaxation Problem

Convex Programming Relaxation (CPR)

Minimize $\hat{f}(x)$

subject to $x(u) + x(v) \geq 1 \quad (\forall e = (u, v) \in E)$

$$x(v) \geq 0 \quad (\forall v \in V)$$

CPR has a half-integral optimal solution.

→ 2-Approximation Algorithm

Submodular Cost Set Cover

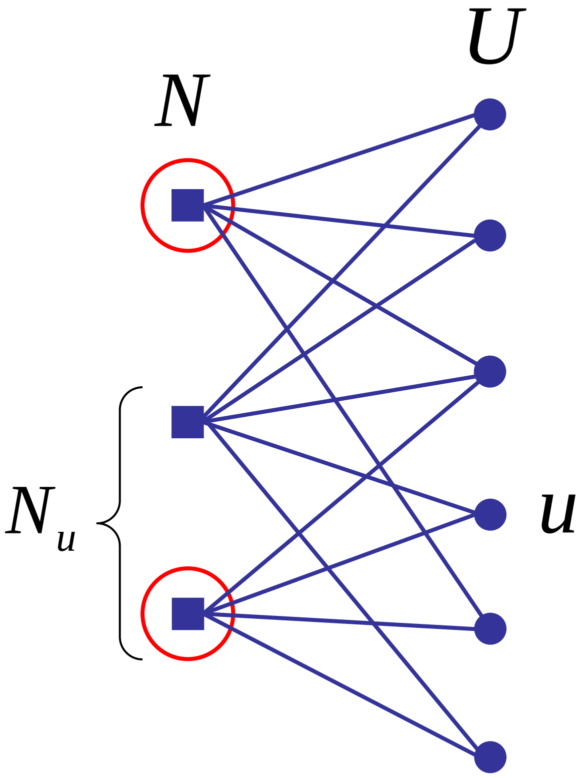
Find $X \subseteq N$ Covering U with Minimum $f(X)$.

$$\eta := \max_{u \in U} |N_u|$$

η-Approximation

Rounding Algorithm

Primal-Dual Algorithm



Summary

- Submodular Functions Arise Everywhere.
- Discrete Analogue of Convexity.
- General SFM Algorithms Available.
- Exploit Special Structures of Problems.
- Approx. Algorithms for Constrained SFM.