A decision procedure for SHOIQ with Transitive Closure of Roles

Chan LE DUC, Myriam LAMOLLE and Olivier CURE

Université Paris 8 - IUT de Montreuil, Université Marne La Vallée

Why transitive closure of roles is needed?

- **Example 1:**

  \[ \textbf{O}_1 : \text{Trans} (\text{reachable}) \]
  \[ \text{click} \sqsubseteq \text{reachable} \]
  \[ A \sqcup B \sqcup C \sqsubseteq \exists \text{reachable}^- . \{ \text{start} \} \]
  \[ C \sqsubseteq \forall \text{click}^- . \bot \]
  \[ \Rightarrow \text{Consistent!} \]

- **Example 2:**

  \[ \text{Human} \text{v9 hasParent} \sqcup \{ \text{Eva} \} \]
  \[ \text{versus} \]
  \[ \text{Human} \text{v9 hasAncestor} \sqcup \{ \text{Eva} \} \]
  \[ \text{where} \text{"hasAncestor"} \text{transitive} \]

\[ \Rightarrow \text{Inconsistent: design error is detected} \]
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Example 1:

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\[ C \sqsubseteq \forall \text{click}^-..\top \]
\[ \Rightarrow \text{Consistent!} \]

Example 2:

\[ \text{O}_2 : A \sqcup B \sqcup C \sqsubseteq \exists (\text{click}^-)^+.\{\text{start}\} \]
\[ C \sqsubseteq \forall \text{click}^-..\bot \]
\[ \Rightarrow \text{Inconsistent : design error is detected} \]

Example 2:

Human $\sqsubseteq \exists \text{hasParent}^+.\{\text{Eva}\}$ versus
Human $\sqsubseteq \exists \text{hasAncestor}.\{\text{Eva}\}$ where “hasAncestor” transitive
The logic $SHOIQ$:
- Finite sets of concept, role and (nominals) individual names
- Concept descriptions:
  $C \sqcap D$, $C \sqcup D$, $\neg C$, $\exists R.C$, $\forall R.C$, $\leq nS.C$, $\geq nS.C$ where $C, D$
  are concepts; $R$ is a role (possibly inverse and transitive); $S$ is a simple role
- Concept axioms: $C \sqsubseteq D$ and role axioms: $R \sqsubseteq S$ : $\text{ontology}$
The logic $\text{SHOIQ}$:
- Finite sets of concept, role and (nominals) individual names
- Concept descriptions:
  $\top, \bot,\neg C, \exists R.C, \forall R.C, \leq nS.C, \geq nS.C$ where $C, D$ are concepts; $R$ is a role (possibly inverse and transitive); $S$ is a simple role
- Concept axioms: $C \sqsubseteq D$ and role axioms: $R \sqsubseteq S$ : ontology

Transitive closure of roles: $(Q^+)^\mathcal{I} = \bigcup_{n>0}(Q^n)^\mathcal{I}$ with an interpretation $\mathcal{I}$

In $\text{SHOIQ}(+)$ ($\text{SHOIQ}$ with transitive closure), one can say: $\exists R^+.C$ or $\forall R^+.C$ but not $\leq nS^+.C$ [Horrocks, Sattler and Tobies, 1999], or $R \sqsubseteq S^+$ [Le Duc and Lamolle, 2010]
Why is $SHOIQ^{(+) \text{ tricky ?}}$

- $SHOIQ$ has a forest-like model whose the infinite part is tree-like [Horrocks and Sattler, 2005]
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- $SHIQ(+)\text{ has a tree-like model [Le Duc, Lamolle and Curé, 2011]}$
Why is $\text{SHOIQ}(+)\text{ tricky?}$

- $\text{SHOIQ}$ has a forest-like model whose the infinite part is tree-like [Horrocks and Sattler, 2005]
- $\text{SHIQ}(+)$ has a tree-like model [Le Duc, Lamolle and Curé, 2011]
- There exists a consistent ontology in $\text{SHOIQ}(+)$ whose all models are non-tree-like

$$\{o\} \subseteq A; A \cap B \subseteq \bot; A \subseteq \exists R.A \cap \exists R'.B; B \subseteq \exists S^+.\{o\}$$

$$\{o\} \subseteq \forall X^- . \bot; X \text{ is functional with } X \in \{R, R', S\}$$

\[\begin{array}{c}
\{o\}, A & R & A & R & A & R & A \\
R', S^- & \quad & R' & \quad & R' & \quad & \\
B, \exists S^+.\{o\} & S^- & B, \exists S^+.\{o\} & S^- & B, \exists S^+.\{o\} & S^- & B, \exists S^+.\{o\}
\end{array}\]
Goal: constructing a model of a $SHO\Omega Q_\mathcal{Q}(+)$ ontology

Key structures:
- A star-type for representing a set of individuals
- A frame and sections for representing a model
- A new blocking condition based on sections

Algorithm:
Tiling a valid frame with star-types until a blocked section is detected
Overview of the algorithm

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Star-type
Star-types and linking

\[ \lambda(\sigma) = l(r'_{0}) \]

\[ r(r_{0}) = r^{-1}(r'_{0}) \]

\[ \lambda(\sigma') = l(r_{0}) \]
Overview of the algorithm

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Algorithm:
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Frame and sections

nominal star-types

\[ o_1 \rightarrow o_2 \rightarrow o_3 \]

a section

\[ \{ \text{diagram} \} \]
nominal star-types

With sections, one can say that a concept $\exists Q^+.C$ is satisfied:
- in the past
- in the future
- somewhere from the future
- somewhere from the past
Overview of the algorithm

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  - A new blocking condition based on sections
- Algorithm:
  Tiling a valid frame with star-types until a blocked section is detected
Each ray \( r \) in the blocking (blocked) section blocks (is blocked by) a ray \( r' \) in the blocked (blocking) section such that (i) \( \mathcal{L}(r) = \mathcal{L}(r') \), and (ii) each \( \exists Q^+.C \) in both \( r \) and \( r' \) is satisfied in "the same way".

Each concept \( \exists Q^+.C \) in the blocking section is satisfied.
Overview of the algorithm

- **Goal**: constructing a model of a $SHIQ(+)\text{ ontology}$
- **Key structures**:
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- **Algorithm**:
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Termination of the algorithm

Termination is a consequence of the following facts:

- The algorithm never removes a star-type
- The number of sections from nominal to blocked one is bounded by $O(2^{2|\mathcal{T}, \mathcal{R}|})$
- Checking satisfaction of a concept $\exists Q^+.D$ over a frame is bounded by a polynomial function in the size of the frame
Soundness: from a valid frame to a model

A frame is valid if

- all star-types are valid
- each nominal star-type is not duplicated
- each concept $\exists Q^+. C$,
  - either it is directly satisfied in the frame
  - or there is a $Q$-sequence leading to the blocked section containing a $\exists Q^+. C$
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Adapting the unravelling technique for frames:

- defining a set of paths over the frame
- extending infinitely the set of defined paths through blocked and blocking sections
- satisfying a concept $\exists Q^+.C$ (extended path) by a $Q$-sequence leading to an extended path or to an initial path containing $\exists Q.C$
Main ideas:

- A model can be reduced to a frame with valid star-types:
  - The first section contains only nominal star-types.
  - It contains a section whose all concepts $\exists Q^+.D$ are satisfied (blocking section).
  - Parts between two “blockable” sections can be removed until a blocked section is detected.

- The reduced model can guide the algorithm to build a valid frame.
Conclusion and Future Work

Conclusion:
- A first decision procedure for $SHOIQ^{(+)}$
- A structure, namely frame, with a new blocking condition for representing infinite non-tree-like parts of a model
- The complexity of the algorithm is high (triply nondeterministic exponential)

Future work:
- Reducing the size of frames
- A more goal-oriented algorithm (tableau algorithm)
- An implementation in progress
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- A first decision procedure for $\text{SHOIQ}^+(\text{+})$
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Future work:
- Reducing the size of frames
- A more goal-oriented algorithm (tableau algorithm)
- An implementation in progress
- Hardness of $\text{SHOIQ}^+(\text{+})$
- The technique could be used for other logics ($\text{ZOIQ}$?)