General Oracle Inequalities for Gibbs Posterior with Application to Ranking

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Our goal is to minimize some theoretical risk of interest \( R \). For example, in bipartite ranking, the probability of misranking

\[
R(\theta) = P[(Y - Y')r(X, X'; \theta) < 0]
\]

\( Y \) is binary in \([-1, 1]\), \( X \) is a \( p \)-dimensional predictor vector, \( r(x, x'; \theta) = 1 \) if \( x \) ranks higher than \( x' \) and \( r(x, x'; \theta) = -1 \) otherwise. \( r \) is parameterized by \( \theta \).

Gibbs posterior is a randomization method of empirical risk minimization

\[
Q(d\theta) = \frac{e^{-\lambda R_n(\theta)} \pi(d\theta)}{\int_{\Theta} e^{-\lambda R_n(\theta)} \pi(d\theta)}
\]

- \( \theta \) is the parameter vector; \( \pi \) is the prior; \( \lambda \) is the inverse temperature parameter.
- \( R_n \) is the empirical risk based on a sample of size \( n \). In the ranking example,

\[
R_n(\theta) = \frac{1}{n(n-1)} \sum_{i \neq j} I[(Y_i - Y_j)r(X_i, X_j; \theta) < 0]
\]

We expect that \( \theta \) sampled from the Gibbs posterior has good theoretical risk performance \( R(\theta) \).
Under model selection, \( \theta = (b, m) \), \( m \) is the model index, \( b \) is the parameter of interest in model space \( B_m \).

The Gibbs posterior can be written as

\[
Q(db, m) = \frac{e^{-\lambda R_n(b)} \pi(db|m) \pi_m}{\sum_m \int_{B_m} e^{-\lambda R_n(b)} \pi(db|m) \pi_m}
\]

Oracle inequality (general form): \( \Delta_m \) is the risk convergence rate on model \( m \).

\[
ER(b) \leq (1 + \delta) \inf_m \left\{ \inf_{b \in B_m} R(b) + O(\Delta_m) \right\}
\]

with \( \theta = (b, m) \) sampled from the Gibbs posterior \( Q \), and small \( \delta \geq 0 \).

- PAC-Bayesian model selection \( \Rightarrow \) Oracle inequalities for Gibbs posterior
  Rigollet and Tsybakov (2011), etc.

- Most of the PAC-Bayesian literature has focused on the additive risk \( R_n(\theta) \),
  with the iid summation form. For example, the iid classification risk with
  classifier \( g \),
  \[
  R_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \left| Y_i - I(g(X_i, \theta) > 0) \right|
  \]

- Our work extends to nonadditive risk \( R_n(\theta) \). For example, the ranking risk
  with ranking rule \( r \),
  \[
  R_n(\theta) = \frac{1}{n(n-1)} \sum_{i \neq j} I[(Y_i - Y_j)r(X_i, X_j; \theta) < 0]
  \]
Main Result: Oracle Inequalities for Gibbs Posterior

$P$ is the true probability measure of the data. $Q$ is the Gibbs posterior measure.

(1) Almost surely in $PQ$-measure, for small $\delta \geq 0$ and all large $n$,

$$R(b) \leq (1 + \delta) \inf_m \left\{ \inf_{b \in B_m} R(b) + O(\Delta_m) \right\};$$

with $(b, m)$ sampled from the Gibbs posterior.

(2) The posterior mean risk can be bounded as

$$E_{PQ} R \leq (1 + \delta) \inf_m \left\{ \inf_{b \in B_m} R(b) + O(\Delta_m) \right\}.$$

Pros and cons of our approach:

+ No assumption on the form of $R_n$;
+ Applicable to the ranking risk and dependent data;
+ Adaptive to the unknown best candidate model;
+ Optimal or near optimal convergence rate;

– Not always strict oracle inequality (leading constant $1 + \delta$ instead of 1).
Application to Bipartite Ranking

Minimal risk $R^*$ is achieved by the $r^*(x, x') = 2I[\eta(x) - \eta(x') > 0] - 1$ where $\eta(x) = P(Y = 1|X = x)$.

We consider linear rules $r(x, x'; b) = 2I[(x - x')^\top b > 0] - 1$ with $b$ in all coordinate subspaces $B_{m,j}$ of $\mathbb{R}^p$, $m = 1, 2, ..., p$ and $j = 0, 1, ..., \binom{p}{m}$.

(1) Almost surely in $PQ$-measure for any $\delta > 0$ and all large $n$, there exists a constant $C_1 > 0$, such that

$$R(b) - R^* \leq (1 + \delta) \inf_{m,j} \left[ \inf_{b \in B_{m,j}} (R(b) - R^*) + \frac{C_1 m (\log n)^3}{n} \right]$$

(2) For any $\delta > 0$ and all large $n$, there exists a constant $C_2 > 0$, such that

$$E_{PQ} R \leq R^* + (1 + \delta) \inf_{m,j} \left[ \inf_{b \in B_{m,j}} (R(b) - R^*) + \frac{C_2 m (\log n)^3}{n} \right]$$

+ Adaptive to the unknown best candidate model;

+ The fast oracle rate of about $O(m/n)$ instead of $O(\sqrt{m/n})$, without assuming the smoothness condition in Clémenton et al. (2008);

+ Dimension $p = o(n/(\log n)^3)$ can increase with the sample size $n$. 
Welcome to my poster for details

Thank You!