adversarial bandit problems: the power of randomization

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on-line prediction

A game between forecaster and environment.

At each round $t$, the forecaster chooses an action $I_t \in \{1, \ldots, N\}$; (actions are often called experts) the environment chooses losses $\ell_t(1), \ldots, \ell_t(N) \in [0, 1]$; the forecaster suffers loss $\ell_t(I_t)$.

The goal is to minimize the average regret $R_n = \frac{1}{n} \left( n \sum_{t=1}^{n} \ell_t(I_t) - \min_{i \leq N} n \sum_{t=1}^{n} \ell_t(i) \right)$. 
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A game between forecaster and environment.

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$$R_n = \frac{1}{n} \left( \sum_{t=1}^{n} \ell_t(l_t) - \min_{i \leq N} \sum_{t=1}^{n} \ell_t(i) \right).$$
Often $\ell_t(i) = \ell(i, y_t)$

where $y_1, \ldots, y_n \in \mathcal{Y}$ is the sequence of outcomes to be predicted.

and $\ell : \{1, \ldots, N\} \times \mathcal{Y} \rightarrow [0, 1]$ is a loss function.
simplest example

Is it possible to make regret $\rightarrow 0$ for all loss assignments?
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$$\ell_t(1) = \begin{cases} 0 & \text{if } I_t = 2 \\ 1 & \text{if } I_t = 1 \end{cases}$$

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Then

$$\sum_{t=1}^{n} \ell_t(I_t) = n \quad \text{and} \quad \min_{i=1,2} \sum_{t=1}^{n} \ell_t(i) \leq \frac{n}{2}$$
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so

$$R_n \geq \frac{1}{2}.$$
randomized prediction

Key to solution: randomization.

At time $t$, the forecaster chooses a probability distribution $p_{t-1} = (p_{1,t-1}, \ldots, p_{N,t-1})$ and chooses action $i$ with probability $p_{i,t-1}$.

Simplest model: all losses $\ell_s(i), i = 1, \ldots, N, s < t$, are observed: full information.
randomized prediction

This and related models have been studied in:
- game theory: playing repeated games;
- information theory: gambling and data compression;
- statistics: sequential decisions;
- statistical learning theory: on-line learning;
Hannan and Blackwell

Hannan (1957) and Blackwell (1956) showed that the forecaster has a strategy such that

$$\frac{1}{n} \left( \sum_{t=1}^{n} \ell_t(l_t) - \min_{i \leq N} \sum_{t=1}^{n} \ell_t(i) \right) \to 0$$

almost surely for all strategies of the environment.
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expected loss of the forecaster:

\[ \ell_t(p_{t-1}) = \sum_{i=1}^{N} p_{i,t-1} \ell_t(i) = \mathbb{E}_t \ell_t(l_t) \]
basic ideas

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By martingale convergence,

\[
\frac{1}{n} \left( \sum_{t=1}^{n} \ell_t(l_t) - \sum_{t=1}^{n} \ell_t(p_{t-1}) \right) = O_P(n^{-1/2})
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basic ideas

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so it suffices to study

\[ \frac{1}{n} \left( \sum_{t=1}^{n} \ell_t(p_{t-1}) - \min_{i \leq N} \sum_{t=1}^{n} \ell_t(i) \right) \]
weighted average prediction

Idea: assign a higher probability to better-performing actions. Vovk (1990), Littlestone and Warmuth (1989).
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A popular choice is

\[
p_{i,t-1} = \frac{\exp \left( -\eta \sum_{s=1}^{t-1} \ell_s(i) \right)}{\sum_{k=1}^{N} \exp \left( -\eta \sum_{s=1}^{t-1} \ell_s(k) \right)} \quad i = 1, \ldots, N.
\]

where \( \eta > 0 \).
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where \( \eta > 0 \). Then

\[ \frac{1}{n} \left( \sum_{t=1}^{n} \ell_t(p_{t-1}) - \min_{i \leq N} \sum_{t=1}^{n} \ell_t(i) \right) = \sqrt{\frac{\ln N}{2n}} \]

with \( \eta = \sqrt{\frac{8 \ln N}{n}} \).
proof

Let \( L_{i,t} = \sum_{s=1}^{t} \ell_s(i) \) and

\[
W_t = \sum_{i=1}^{N} w_{i,t} = \sum_{i=1}^{N} e^{-\eta L_{i,t}}
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for \( t \geq 1 \), and \( W_0 = N \).
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for \( t \geq 1 \), and \( W_0 = N \). First observe that

\[
\ln \frac{W_n}{W_0} = \ln \left( \sum_{i=1}^{N} e^{-\eta L_{i,n}} \right) - \ln N
\]

\[
\geq \ln \left( \max_{i=1,\ldots,N} e^{-\eta L_{i,n}} \right) - \ln N
\]

\[
= -\eta \min_{i=1,\ldots,N} L_{i,n} - \ln N.
\]
proof

On the other hand, for each \( t = 1, \ldots, n \)

\[
\ln \frac{W_t}{W_{t-1}} = \ln \frac{\sum_{i=1}^{N} w_{i,t-1} e^{-\eta \ell_t(i)}}{\sum_{j=1}^{N} w_{j,t-1}} \\
\leq -\eta \frac{\sum_{i=1}^{N} w_{i,t-1} \ell_t(i)}{\sum_{j=1}^{N} w_{j,t-1}} + \frac{\eta^2}{8} \\
= -\eta \ell_t(p_{t-1}) + \frac{\eta^2}{8}
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by Hoeffding’s inequality.
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by Hoeffding’s inequality.

Hoeffding (1963): if \( X \in [0, 1] \),

\[
\ln \mathbb{E}e^{-\eta X} \leq -\eta \mathbb{E}X + \frac{\eta^2}{8}
\]
proof

for each $t = 1, \ldots, n$

$$\ln \frac{W_t}{W_{t-1}} \leq -\eta \ell_t(p_{t-1}) + \frac{\eta^2}{8}$$

Summing over $t = 1, \ldots, n$,

$$\ln \frac{W_n}{W_0} \leq -\eta \sum_{t=1}^n \ell_t(p_{t-1}) + \frac{\eta^2}{8} n.$$

Combining these, we get

$$\sum_{t=1}^n \ell_t(p_{t-1}) \leq \min_{i=1,\ldots,N} L_{i,n} + \frac{\ln N}{\eta} + \frac{\eta}{8} n.$$
lower bound

The upper bound is optimal: for all predictors,

$$\sup_{n,N,\ell_t(i)} \frac{\sum_{t=1}^n \ell_t(I_t) - \min_{i \leq N} \sum_{t=1}^n \ell_t(i)}{\sqrt{(n/2) \ln N}} \geq 1.$$
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Idea: choose \( \ell_t(i) \) to be i.i.d. symmetric Bernoulli coin flips.
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\[
\sup_{\ell_t(i)} \left( \sum_{t=1}^{n} \ell_t(I_t) - \min_{i \leq N} \sum_{t=1}^{n} \ell_t(i) \right)
\geq \mathbb{E} \left[ \sum_{t=1}^{n} \ell_t(I_t) - \min_{i \leq N} \sum_{t=1}^{n} \ell_t(i) \right]
= \frac{n}{2} - \min_{i \leq N} B_i
\]

Where \( B_1, \ldots, B_N \) are independent Binomial \( (n, 1/2) \).
Use the central limit theorem.
The forecaster does not see the outcomes $\ell_t(i)$ unless he asks for them, but can do it only $m \ll n$ times.
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- the forecaster incurs loss $\ell_t(l_t)$ and each action $i$ incurs loss $\ell_t(i)$. not of revealed to the forecaster!
efficient prediction

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- the forecaster incurs loss $\ell_t(l_t)$ and each action $i$ incurs loss $\ell_t(i)$. not of revealed to the forecaster!;
- the forecaster decides whether he asks for the values of $\ell_t(i)$ if the total number of revealed outcomes up to time $t - 1$ is less than $m$. 
a label efficient forecaster

Idea: ask for values randomly (with probability $\approx \frac{m}{n}$) and use the weighted average forecaster with the estimated losses.
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The forecaster asks for \(\ell_t(i)\) iff \(Z_t = 1\). Let

\[
\tilde{\ell}_t(i) \overset{\text{def}}{=} \begin{cases} 
\ell_t(i)/\epsilon & \text{if } Z_t = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

An unbiased estimate!
a label efficient forecaster

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An unbiased estimate!

For each round $t = 1, 2, \ldots, n$ draw an action from $\{1, \ldots, N\}$ according to the distribution

$$p_{i,t-1} = \frac{\exp\left(-\eta \sum_{s=1}^{t-1} \tilde{\ell}_s(i)\right)}{\sum_{k=1}^{N} \exp\left(-\eta \sum_{s=1}^{t-1} \tilde{\ell}_s(k)\right)} \quad i = 1, \ldots, N.$$
bound for label efficient prediction

With probability at least $1 - \delta$,

$$\frac{1}{n} \left( \sum_{t=1}^{n} \ell_t(I_t) - \min_{i \leq N} \sum_{t=1}^{n} \ell_t(i) \right) \leq 9 \sqrt{\frac{\ln N + \ln(4/\delta)}{m}}.$$  

(Cesa-Bianchi, Lugosi, Stoltz, 2005)
bound for label efficient prediction

With probability at least $1 - \delta$,

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Sketch of proof: First bound

$$\sum_{t=1}^{n} \tilde{\ell}_t(p_{t-1}) - \min_{i \leq N} \sum_{t=1}^{n} \tilde{\ell}_t(i)$$

as before.
With probability at least $1 - \delta$, 

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Sketch of proof: First bound

$$
\sum_{t=1}^{n} \tilde{\ell}_t(p_{t-1}) - \min_{i \leq N} \sum_{t=1}^{n} \tilde{\ell}_t(i)
$$

as before. Then use Bernstein-type martingale inequalities to handle

$$
\sum_{t=1}^{n} \ell_t(l_t) - \sum_{t=1}^{n} \tilde{\ell}_t(p_{t-1})
$$

and

$$
\min_{i \leq N} \sum_{t=1}^{n} \tilde{\ell}_t(i) - \min_{i \leq N} \sum_{t=1}^{n} \ell_t(i)
$$
For any forecaster asking for at most $m$ values,

$$\sup_{\ell_t(i) \in \{0,1\}} \frac{1}{n} \left( \sum_{t=1}^{n} \ell_t(l_t) - \min_{i \leq N} \sum_{t=1}^{n} \ell_t(i) \right) \geq C \sqrt{\frac{\ln N}{m}}.$$
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\]

**Idea (for $N = 2$):** choose the losses randomly (i.i.d.) such that they are either Bernoulli $1/2 - \epsilon$ or Bernoulli $1/2 + \epsilon$. 
multi-armed bandits

The forecaster only observes $\ell_t(l_t)$ but not $\ell_t(i)$ for $i \neq l_t$. 

Herbert Robbins (1952).
The forecaster only observes $\ell_t(l_t)$ but not $\ell_t(i)$ for $i \neq l_t$.

Herbert Robbins (1952).
multi-armed bandits

a one-armed bandit
multi-armed bandits

Compulsive gambling

a multi-armed bandit
multi-armed bandits

Trick: estimate $\ell_t(i)$ by

$$\tilde{\ell}_t(i) = \frac{\ell_t(l_t) \mathbb{1}_{\{l_t=i\}}}{p_{l_t, t-1}}$$
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$$\mathbb{E}_t \tilde{\ell}_t(i) = \sum_{j=1}^{N} p_{j,t-1} \frac{\ell_t(j) \mathbb{1}_{\{j=i\}}}{p_{j,t-1}} = \ell_t(i)$$
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Use the estimated losses to define exponential weights and mix with uniform (Auer, Cesa-Bianchi, Freund, and Schapire, 2002):

$$p_{i,t-1} = (1 - \gamma) \frac{\exp \left( -\eta \sum_{s=1}^{t-1} \tilde{\ell}_s(i) \right)}{\sum_{k=1}^{N} \exp \left( -\eta \sum_{s=1}^{t-1} \tilde{\ell}_s(k) \right)} + \frac{\gamma}{N}$$
multi-armed bandits

\[
\mathbb{E}\left(\frac{1}{n} \left( \sum_{t=1}^{n} \ell_t(p_{t-1}) - \min_{i \leq N} \sum_{t=1}^{n} \ell_t(i) \right) \right) = O\left(\sqrt{\frac{N \ln N}{n}}\right),
\]
multi-armed bandits

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multi-armed bandits

Lower bound:

$$\sup_{\ell_t(i)} \frac{1}{n} \left( \sum_{t=1}^{n} \ell_t(p_{t-1}) - \min_{i \leq N} \sum_{t=1}^{n} \ell_t(i) \right) \geq C \sqrt{\frac{N}{n}},$$

Dependence on $N$ is not logarithmic anymore!
multi-armed bandits

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Dependence on \( N \) is not logarithmic anymore!

Audibert and Bubeck (2009) constructed a forecaster with

\[
\max_{i \leq N} \frac{1}{n} \left( \sum_{t=1}^{n} \ell_t(p_{t-1}) - \sum_{t=1}^{n} \ell_t(i) \right) = O \left( \sqrt{\frac{N}{n}} \right),
\]
follow the perturbed leader

$$l_t = \arg \min_{i=1,\ldots,N} \sum_{s=1}^{t-1} \ell_s(i) + Z_{i,t}$$

where the $Z_{i,t}$ are random noise variables.

By carefully defining the distribution of $Z_{i,t}$ one can get similar regret bounds for the full information case, Hannan (1957); Kalai and Vempala (2003).
combinatorial experts

Often the class of experts is very large but has some combinatorial structure. Can the structure be exploited?
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examples:

path planning. At each time instance, the forecaster chooses a path in a graph between two fixed nodes. Each edge has an associated loss. Loss of a path is the sum of the losses over the edges in the path.

N is huge!!!
Given a complete bipartite graph $K_{m,m}$, the forecaster chooses a perfect matching. The loss is the sum of the losses over the edges.

The forecaster chooses a spanning tree in the complete graph $K_m$. The cost is the sum of the losses over the edges.
Two models.

**(Easy.)** Losses of the components of the chosen object are observed separately. (György, Lugosi, Ottucsák, 2007.)

**(Interesting.)** Only total loss of the chosen object is observed. (Awerbuch and Kleinberg, 2004; McMahan and Blum, 2004; Dani, Hayes, and Kakade, 2008; Abernethy, Hazan, and Rakhlin, 2008; Bartlett, Dani, Hayes, Kakade, and Tewari, 2008; Cesa-Bianchi and Lugosi, 2009.)
Performance bounds: Is $O(n^{-1/2} \text{poly}(d))$ regret achievable for the bandit problem?

Algorithmic: How can one draw a random object from the exponentially weighted distribution in polynomial time?
\( S = \{v_1, \dotsc, v_N\} \subset \mathbb{R}^d \) is a collection of objects (experts).

Denote \( B = \max_{v_k} \|v_k\|_2 \).

At every time \( t = 1, 2, \dotsc \), the opponent chooses a loss vector \( \ell_t \in \mathbb{R}^d \).

We assume \( \ell_t(k) = \ell_t^T v_k \in [-1, 1] \) for all \( v_k \in S \).
linear bandit problem

For $t = 1, 2, \ldots$,
- opponent chooses $\ell_t \in \mathbb{R}^d$
- Forecaster chooses $K_t \in \{1, \ldots, N\}$
- The cost $\ell_t(K_t) = \ell_t^T v_{K_t}$ is revealed.

The forecaster’s goal is to control the expected regret

$$\mathbb{E}\hat{L}_n - \min_{k=1, \ldots, N} L_n(k) = \sum_{t=1}^n \mathbb{E}\ell_t(K_t) - \min_{k=1, \ldots, N} \sum_{t=1}^n \ell_t(k).$$

Expectation is with respect to the forecaster’s internal randomization.
weighted average forecaster

At time $t$ assign a weight $w_{t,i}$ to each $i = 1, \ldots, d$. The weight of each $v_k \in S$ is

$$
\overline{w}_t(k) = \prod_{i: v_k(i)=1} w_{t,i}.
$$
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$$w_t(k) = \prod_{i : v_k(i) = 1} w_{t,i}.$$ 

Let $q_{t-1}(k) = w_{t-1}(k) / \sum_{k=1}^{N} w_{t-1}(k)$. 
weighted average forecaster

At time $t$ assign a weight $w_{t,i}$ to each $i = 1, \ldots, d$.

The weight of each $v_k \in S$ is

$$w_t(k) = \prod_{i: v_k(i) = 1} w_{t,i} .$$

Let $q_{t-1}(k) = w_{t-1}(k) / \sum_{k=1}^{N} w_{t-1}(k)$.

At each time $t$, draw $K_t$ from the distribution

$$p_{t-1}(k) = (1 - \gamma) q_{t-1}(k) + \gamma \mu(k)$$

where $\mu$ is a fixed distribution on $S$ and $\gamma > 0$. Here

$$w_{t,i} = \exp(-\eta \tilde{L}_{t,i})$$

where $\tilde{L}_{t,i} = \tilde{\ell}_{1,i} + \cdots + \tilde{\ell}_{t,i}$ and $\tilde{\ell}_{t,i}$ is an estimated loss.
Define the scaled incidence vector

\[ X_t = \ell_t(K_t) V_{K_t} \]

where \( K_t \) is distributed according to \( p_{t-1} \).
loss estimates


Define the scaled incidence vector

\[ X_t = \ell_t(K_t)V_{K_t} \]

where \( K_t \) is distributed according to \( p_{t-1} \).

Let \( P_{t-1} = \mathbb{E}[V_{K_t}V_{K_t}^\top] \) be the \( d \times d \) correlation matrix. Hence

\[ P_{t-1}(i, j) = \sum_{k : v_k(i) = v_k(j) = 1} p_{t-1}(k) . \]

Similarly, let \( Q_{t-1} \) and \( M \) be the correlation matrices of \( \mathbb{E}[V V^\top] \) when \( V \) has law, \( q_{t-1} \) and \( \mu \). Then

\[ P_{t-1}(i, j) = (1 - \gamma)Q_{t-1}(i, j) + \gamma M(i, j) . \]
Define the scaled incidence vector

\[ X_t = \ell_t(K_t)V_{K_t} \]

where \( K_t \) is distributed according to \( p_{t-1} \).

Let \( P_{t-1} = \mathbb{E}[V_{K_t} V_{K_t}^T] \) be the \( d \times d \) correlation matrix.

Hence

\[ P_{t-1}(i, j) = \sum_{k : v_k(i)=v_k(i)=1} p_{t-1}(k) . \]

Similarly, let \( Q_{t-1} \) and \( M \) be the correlation matrices of \( \mathbb{E}[V V^T] \) when \( V \) has law, \( q_{t-1} \) and \( \mu \). Then

\[ P_{t-1}(i, j) = (1 - \gamma)Q_{t-1}(i, j) + \gamma M(i, j) . \]

The vector of loss estimates is defined by

\[ \tilde{\ell}_t = P_{t-1}^+ X_t \]

where \( P_{t-1}^+ \) is the pseudo-inverse of \( P_{t-1} \).
key properties

- $M M^+ v = v$ for all $v \in S$.
- $Q_{t-1}$ is positive semidefinite for every $t$.
- $P_{t-1} P_{t-1}^+ v = v$ for all $t$ and $v \in S$.

By definition,

$$E_t X_t = P_{t-1} \ell_t$$

and therefore

$$E_t \tilde{\ell}_t = P_{t-1}^+ E_t X_t = \ell_t$$

An unbiased estimate!
The regret of the forecaster satisfies

$$\frac{1}{n} \left( \mathbb{E} \hat{L}_n - \min_{k=1,\ldots,N} L_n(k) \right) \leq 2 \sqrt{\left( \frac{2B^2}{d\lambda_{\min}(M)} + 1 \right) \frac{d \ln N}{n}}.$$

where

$$\lambda_{\min}(M) = \min_{x \in \text{span}(S): \|x\|=1} x^T M x > 0$$

is the smallest “relevant” eigenvalue of $M$. (Cesa-Bianchi and Lugosi, 2009.)

Large $\lambda_{\min}(M)$ is needed to make sure no $|\tilde{\ell}_{t,i}|$ is too large.
performance bound

Other bounds:

\[ B \sqrt{d \ln N / n} \] (Dani, Hayes, and Kakade). No condition on \( S \).
Sampling is over a barycentric spanner.

\[ d \sqrt{(\theta \ln n) / n} \] (Abernethy, Hazan, and Rakhlin). Computationally efficient.
eigenvalue bounds

\[ \lambda_{\min}(M) = \min_{x \in \text{span}(S): \|x\|=1} \mathbb{E}(V, x)^2. \]

where \( V \) has distribution \( \mu \) over \( S \).

In many cases it suffices to take \( \mu \) uniform.
multitask bandit problem

The decision maker acts in \( m \) games in parallel. In each game, the decision maker selects one of \( R \) possible actions. After selecting the \( m \) actions, the sum of the losses is observed.

\[
\lambda_{\text{min}} = \frac{1}{R}
\]

\[
\max_k \mathbb{E} \left[ \hat{L}_n - L_n(k) \right] \leq 2m\sqrt{3nR \ln R}.
\]

The price of only observing the sum of losses is a factor of \( m \).

Generating a random joint action can be done in polynomial time.
Perfect matchings of $K_{m,m}$. At each time one of the $N = m!$ perfect matchings of $K_{m,m}$ is selected.

$$\lambda_{\text{min}}(M) = \frac{1}{m - 1}$$

$$\max_k \mathbb{E} \left[ \hat{L}_n - L_n(k) \right] \leq 2m \sqrt{3n \ln(m!)}.$$ 

Only a factor of $m$ worse than naive full-information bound.

Sum of weights (partition function) is the permanent of a non-negative matrix. Sampling may be done by a FPAS of Jerrum, Sinclair, and Vigoda (2004).
spanning trees

In a network of $m$ nodes, the cost of communication between two nodes joined by edge $e$ is $\ell_t(e)$ at time $t$. At each time a minimal connected subnetwork (a spanning tree) is selected. The goal is to minimize the total cost. $N = m^{m-2}$.

$$\lambda_{\text{min}}(M) = \frac{1}{m} - O \left( \frac{1}{m^2} \right).$$

The entries of $M$ are

$$\mathbb{P}\{V_i = 1\} = \frac{2}{m},$$

$$\mathbb{P}\{V_i = 1, V_j = 1\} = \frac{3}{m^2} \quad \text{if } i \sim j,$$

$$\mathbb{P}\{V_i = 1, V_j = 1\} = \frac{4}{m^2} \quad \text{if } i \not\sim j.$$
Propp and Wilson (1998) define an exact sampling algorithm. Expected running time is the average hitting time of the Markov chain defined by the edge weights \( w_t(e) = \exp(-\eta \tilde{L}_t(e)) \).
stars

At each time a central node of a network of $m$ nodes is selected. Cost is the total cost of the edges adjacent to the node.

$$\lambda_{\text{min}} \geq 1 - O\left(\frac{1}{m}\right).$$
A balanced cut in $K_{2m}$ is the collection of all edges between a set of $m$ vertices and its complement. Each balanced cut has $m^2$ edges and there are $N = \binom{2m}{m}$ balanced cuts.

$$\lambda_{\text{min}}(M) = \frac{1}{4} - O\left(\frac{1}{m^2}\right).$$

Choosing from the exponentially weighted average distribution is equivalent to sampling from ferromagnetic Ising model. FPAS by Randall and Wilson (1999).
A Hamiltonian cycle in $K_m$ is a cycle that visits each vertex exactly once and returns to the starting vertex. $N = (m - 1)!$

$$\lambda_{\text{min}} \geq \frac{2}{m}$$

Efficient computation is hopeless.
sampling paths

In all these examples $\mu$ is uniform over $S$.

For path planning it does not always work.
sampling paths

In all these examples $\mu$ is uniform over $\mathcal{S}$.

For path planning it does not always work.

What is the optimal choice of $\mu$?
What is the optimal way of exploration?
For each round $t = 1, \ldots, n$,

- the environment chooses the next outcome $J_t \in \{1, \ldots, M\}$ without revealing it;
- the forecaster chooses a probability distribution $p_t$ and draws an action $I_t \in \{1, \ldots, N\}$ according to $p_t$;
- the forecaster incurs loss $\ell(I_t, J_t)$ and each action $i$ incurs loss $\ell(i, J_t)$. None of these values is revealed to the forecaster;
- the feedback $h(I_t, J_t)$ is revealed to the forecaster.

$H = [h(i, j)]_{N \times M}$ is the feedback matrix.

$L = [\ell(i, j)]_{N \times M}$ is the loss matrix.
Dynamic pricing. Here $M = N$, and $L = [\ell(i, j)]_{N \times N}$ where

$$\ell(i, j) = \frac{(j - i) \mathbb{1}_{\{i \leq j\}} + c \mathbb{1}_{\{i > j\}}}{N}.$$ 

and $h(i, j) = \mathbb{1}_{\{i > j\}}$ or

$$h(i, j) = a \mathbb{1}_{\{i \leq j\}} + b \mathbb{1}_{\{i > j\}}, \quad i, j = 1, \ldots, N.$$
**Dynamic pricing.** Here $\mathbf{M} = \mathbf{N}$, and $\mathbf{L} = [\ell(i, j)]_{\mathbf{N} \times \mathbf{N}}$ where

$$\ell(i, j) = \frac{(j - i) \mathbb{1}_{\{i \leq j\}} + c \mathbb{1}_{\{i > j\}}}{N} .$$

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$$h(i, j) = a \mathbb{1}_{\{i \leq j\}} + b \mathbb{1}_{\{i > j\}} , \quad i, j = 1, \ldots, N .$$

**Multi-armed bandit problem.** The only information the forecaster receives is his own loss: $\mathbf{H} = \mathbf{L}$. 

examples
examples

Apple tasting.  \( N = M = 2 \).

\[
L = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

\[
H = \begin{bmatrix}
a & a \\
b & c
\end{bmatrix}.
\]

The predictor only receives feedback when he chooses the second action.
examples

Apple tasting. \( N = M = 2 \).

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\[
H = \begin{bmatrix} a & a \\ b & c \end{bmatrix}.
\]

The predictor only receives feedback when he chooses the second action.

Label efficient prediction. \( N = 3, M = 2 \).

\[
L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

\[
H = \begin{bmatrix} a & b \\ c & c \\ c & c \end{bmatrix}.
\]
a general predictor

A forecaster first proposed by Piccolboni and Schindelhauer (2001). Crucial assumption: \( H \) can be encoded such that there exists an \( N \times N \) matrix \( K = [k(i, j)]_{N \times N} \) such that

\[
L = K \cdot H.
\]

Thus,

\[
\ell(i, j) = \sum_{l=1}^{N} k(i, l)h(l, j).
\]

Then we may estimate the losses by

\[
\tilde{\ell}(i, J_t) = \frac{k(i, I_t)h(I_t, J_t)}{p_{I_t, t}}.
\]
a general predictor

Observe

\[
\mathbb{E}_t \tilde{\ell}(i, J_t) = \sum_{k=1}^{N} p_{k,t} \frac{k(i, k)h(k, J_t)}{p_{k,t}}
\]

\[
= \sum_{k=1}^{N} k(i, k)h(k, J_t) = \ell(i, J_t)
\]

\(\tilde{\ell}(i, J_t)\) is an unbiased estimate of \(\ell(i, J_t)\).

Let

\[
p_{i,t} = (1 - \gamma) \frac{e^{-\eta \tilde{L}_{i,t-1}}}{\sum_{k=1}^{N} e^{-\eta \tilde{L}_{k,t-1}}} + \frac{\gamma}{N}
\]

where \(\tilde{L}_{i,t} = \sum_{s=1}^{t} \tilde{\ell}(i, J_t)\).
performance bound

With probability at least $1 - \delta$,

$$\frac{1}{n} \sum_{t=1}^{n} \ell(I_t, J_t) - \min_{i=1, \ldots, N} \frac{1}{n} \sum_{t=1}^{n} \ell(i, J_t) \leq C n^{-1/3} N^{2/3} \sqrt{\ln(N/\delta)} .$$

where $C$ depends on $K$. (Cesa-Bianchi, Lugosi, Stoltz (2006))
performance bound

With probability at least $1 - \delta$, 

$$\frac{1}{n} \sum_{t=1}^{n} \ell(I_t, J_t) - \min_{i=1,\ldots,N} \frac{1}{n} \sum_{t=1}^{n} \ell(i, J_t) \leq C n^{-1/3} N^{2/3} \sqrt{\ln(N/\delta)}.$$

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Hannan consistency is achieved with rate $O(n^{-1/3})$ whenever $L = K \cdot H$.

This solves the dynamic pricing problem.
performance bound

With probability at least $1 - \delta$,

$$\frac{1}{n} \sum_{t=1}^{n} \ell(I_t, J_t) - \min_{i=1, \ldots, N} \frac{1}{n} \sum_{t=1}^{n} \ell(i, J_t) \leq Cn^{-1/3}N^{2/3} \sqrt{\ln(N/\delta)}.$$ 

where $C$ depends on $K$. (Cesa-Bianchi, Lugosi, Stoltz (2006))

Hannan consistency is achieved with rate $O(n^{-1/3})$ whenever $L = K \cdot H$.

This solves the dynamic pricing problem.

Bartók, Pál, and Szepesvári (2010): if $M = 2$, only possible rates are $n^{-1/2}, n^{-1/3}, 1$
imperfect monitoring: a general framework

$S$ is a finite set of signals.

Feedback matrix: $H : \{1, \ldots, N\} \times \{1, \ldots, M\} \rightarrow \mathcal{P}(S)$.

For each round $t = 1, 2, \ldots, n$,

- the environment chooses the next outcome $J_t \in \{1, \ldots, M\}$ without revealing it;
- the forecaster chooses $p_t$ and draws an action $I_t \in \{1, \ldots, N\}$ according to it;
- the forecaster receives loss $\ell(I_t, J_t)$ and each action $i$ suffers loss $\ell(i, J_t)$, none of these values is revealed to the forecaster;
- a feedback $s_t$ drawn at random according to $H(I_t, J_t)$ is revealed to the forecaster.
Define

\[ \ell(p, q) = \sum_{i,j} p_i q_j \ell(i, j) \]

\[ H(\cdot, q) = (H(1, q), \ldots, H(N, q)) \]

where \( H(i, q) = \sum_j q_j H(i, j) \).

Denote by \( \mathcal{F} \) the set of those \( \Delta \) that can be written as \( H(\cdot, q) \) for some \( q \).

\( \mathcal{F} \) is the set of “observable” vectors of signal distributions \( \Delta \).

The key quantity is

\[ \rho(p, \Delta) = \max_{q : H(\cdot, q) = \Delta} \ell(p, q) \]

\( \rho \) is convex in \( p \) and concave in \( \Delta \).
The value of the base one-shot game is

\[
\min_p \max_q \ell(p, q) = \min_p \max_{\Delta \in \mathcal{F}} \rho(p, \Delta)
\]

If \( \bar{q}_n \) is the empirical distribution of \( J_1, \ldots, J_n \), even with the knowledge of \( H(\cdot, \bar{q}_n) \) we cannot hope to do better than \( \min_p \rho(p, H(\cdot, \bar{q}_n)) \).

Rustichini (1999) proved that there exists a strategy such that for all strategies of the opponent, almost surely,

\[
\limsup_{n \to \infty} \left( \frac{1}{n} \sum_{t=1, \ldots, n} \ell(I_t, J_t) - \min_p \rho(p, H(\cdot, \bar{q}_n)) \right) \leq 0
\]
Rustichini’s proof relies on an approachability theorem for a continuum of types (Mertens, Sorin, and Zamir, 1994).

It is non-constructive.

It does not imply any convergence rate.