BAYESIAN NUMERICAL ANALYSIS

CONSIDER: \( f(x) = e^{\cosh\left(\frac{x^2 + 5}{\cos(x) + 3}\right)} \)

WHAT IS \( \int f(x) \, dx \)?

WHAT DOES IT MEAN TO "KNOW A FUNCTION"?

IDEA: ADMIT DON'T KNOW \( f \), PUT A PRIOR ON \( f(x) \)

SAY \( f(x) = A + CB(x) \) (BROWNIAN MOTION)

OBSERVE: \( y_1 = f(x_1), \ldots, y_n = f(x_n) \)

\( f \sim A + CB(x) \) CONDITIONED \( y_i \) AT \( x_i \)

BAYES RULE FOR \( f \) IS STRAIGHT LINE INTERPOLANT

\( \hat{f} \) IS TRAPEZOID RULE
Wait this $f$ is smooth; B.M. not my prior!

Ok, use $f(x) \sim \int_0^x \text{kernel}$, get cubic splines

or, integrate $k$ times; $2k+1$ splines

Maybe this Bayesian approach is not so crazy

Program: take standard numerical tasks

(Quadrature, interpolation, roots, maxima, ...)

and see if they have Bayesian interpretation

Good

- Possible in high dimensions
- Gives posterior, not just Bayes rule
- Gives a different way of thinking about
  N.A. (versus, 'h^th order', '# operations')
QUESTIONS

• IS SIMPSONS RULE BAYES? (NOPE)

• IS B.M. THE ONLY PRIOR GIVING TRAPEZOID RULE?
  • POISSON PROCESS (ANY INDEPENDENT INCREMENT)
  • C_{B(t)} PREDICTS SAME, C \perp B(t)

THEOREM (WILLIAMS) IF X_t ON [0,1] HAS CONTINUOUS SAMPLE PATHS AND PREDICTS LIKE B.M., THEN

X_t \sim C_{B(t)} \perp B(t)

WHY X_t IS A MARTINGALE
USE LEVY'S CHARACTERIZATION
HISTORY

H. POINCARÉ (1896)

\( f(x) \) UNKNOWN ON IR

GIVEN \( f(x_i) = y_i \), \( 1 \leq i \leq N \)

ESTIMATE \( f(x) \) (INTERPOLATION)

IDEA: ASSUME \( f(x) = \sum_{j=0}^{\infty} A_j x^j \)

\( A_j \sim \mathcal{N}(0, \sigma_j^2) \) so \( f(x) \) IS GAUSSIAN

\( \sum_{j=0}^{\infty} \sigma_j^2 x^j \)

So \( f(x_1, \ldots, x_N) \) IS GAUSSIAN

IF \( \sigma(x) = \sum_{j=0}^{\infty} \sigma_j^2 x^j \)

\( \hat{f}(x) = a_0 \sigma(x_1) + \ldots + a_N \sigma(x_N) \)

\( a_i \) CHOSEN SO \( \hat{f}(x_i) = y_i \)

"I USE THE METHOD OF CAUSES"
WHATS NEW?

LOTS OF APPLICATIONS IN FUNCTION SPACE SETTING

A. STUART (2010) INVERSE PROBLEMS: A BAYESIAN APPROACH. ACTA NUMERICA

\[ y = g(u) + \epsilon \]

PROBLEM ESTIMATE U

HERE y AND U ARE FUNCTIONS

EX (STOKES EQUATION)

\[ \frac{\partial v}{\partial t} = \nu \Delta v - \nabla p + f, \ (x,t) \in Q \times (0,\omega) \]
\[ \nabla \cdot v = 0, \quad v = u \text{ at } (\theta, 0) \]

OBSEERVE TRACES \[ z_j(t) = \int_0^1 v(x, y(t), t) \, dx \]
AT \( x_i \) \( 1 \leq i \leq 1 \leq j \leq 5 \) THIS IS Y

OBJECT ESTIMATE U GIVEN \( z_j(x_i) \)
LOTS OF USEFUL NEW PRIORS

**Problem:** Given a set $\mathbf{X}$, find useful natural priors indexed by $\mathbf{X}$

$$\mathbf{X} \ni \mathbf{x} = \mathbb{Z}_d^{d - 2}$$

$f(\mathbf{x})$ measures 'throughput'

Find $\max_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

With a prior, can choose informative observations sequentially

**Idea (Ylvisaker):** Use Dynkins isomorphism:

**Easy to make (reversible) Markov chain**

$\pi(x), K(x,y)$ (i.e., $\pi(x)K(x,y) = \pi(y)K(y,x)$) indexed by $\mathbf{X}$

$$G(x,y) = \sum_{n=0}^{\infty} K^n(x,y) \quad (\text{need } K \text{ absorbing})$$

$G(x,y)$ is p.d. so the covariance of a Gaussian field
E(X_v | X_a, s ∈ s) = ∑ a∈s C_a X_a

C_a = P{ X_y, started at x, first hits s in t}

Point: C_a ≥ 0, covariances positive.
There is a converse.

Point: if the underlying Markov chain is
N.N. random walk on a graph,
then X_v is the Gaussian free field.
An object of intense study in modern
probability.

Point: there are now Gaussian analogs.
STEVE EVELNS AND I HAVE A DUAL CONSTRUCTION WHICH GIVES NEGATIVE COVARIANCES (AND ALL SORTS OF NICE PROPERTIES):

1. Let \( \pi(y), k(x,y) \) be reversible on \( \text{finite} \) \( X \).
2. Form a graph on \( X \) with \( y \sim y \leq k(x,y) > 0 \).
3. Choose an orientation \( \epsilon_{xy} = \pm 1 \) each edge.
4. Put a mean 0, var \( \text{Var}(k(y)) \) Gaussian on each edge \( z(x,y) \).
5. Set \( z_x = \sum_y \epsilon_{xy} z_{xy} \).
6. Then \( \{z_y\}_{y \in X} \) has covariances \( \Sigma(x,y) = \begin{cases} \pi(y) & y = x \\ -\pi(y)k(x,y) & \end{cases} \).
7. If \( k(x,y) = 0 \) \( \text{cov}(z_x, z_y) = 0 \).

\[
\begin{align*}
z_1 &= z_{51} - z_{12} \\
z_2 &= z_{12} - z_{23} \\
z_3 &= z_{23} - z_{31} \\
z_4 &= z_{31} - z_{12} \\
z_5 &= z_{12} - z_{23} \\
\end{align*}
\]

WITH KHANE, WE HAVE SIMILAR PROCESS WITH GENERAL SIGN PATTERN, ON GENERAL SPACES
Figure 2.1: Signs of realizations of various Gaussian fields for a 30 × 30 grid (with a different sign matrix) (a) Dynkin’s construction (b) Generalized Dynkin’s construction (c) Diaconis-Evans’ construction (d) Generalized Diaconis-Evans’ construction.
QUESTIONS

1. IS THERE A PRIOR GIVING EVEN ORDER SPLINES (THE HALF INTEGRAL DOESN’T WORK)

2. CHARACTERIZE \( \mathcal{B}(\xi_1) \sim \int_0^{\xi_1} \mathcal{B}(\xi) \) A LA HAMEL’S

3. MORE GENERALLY, WHEN DOES THE BAYES RULE CHARACTERIZE THE PRIOR?

4. DESIGN: USE PRIOR TO SAY WHERE TO TAKE OBSERVATIONS.
   
   \[ \text{eq. } \mathcal{B}(\xi_1) \sim \mathcal{B}(\xi) \text{ ON } [0, \xi_1] \text{ TO ESTIMATE } I = \int_0^{\xi_1} \mathcal{B}(\xi) \, d\xi. \]

   BEST \( n \) POINTS ARE \( \frac{2i}{2n+1} \leq \xi \leq \frac{2i+1}{2n+1} \)

   \( \text{eq. } n=2, \frac{2}{3}, \frac{4}{3} \)

5. TRY THESE THINGS OUT (IT’S EASIER TO PROVE THEOREMS THAN TO DO SERIOUS EXAMPLES)