Erdős-Ko-Rado Theorems

Chris Godsil

WaterMellon, May 2009

Collaborators

Mike Newman, Karen Meagher

Outline

1 Erdős-Ko-Rado
- The Theorem
- Sets to Graphs

2 A Method
- Bounds
- Equality

3 Characterisations
- Kneser
- Derangements

Outline

1. Erdős-Ko-Rado
   - The Theorem
   - Sets to Graphs
2. A Method
   - Bounds
   - Equality
3. Characterisations
   - Kneser
   - Derangements

Intersecting Families

Definition

A family of subsets $\mathcal{F}$ of some set is intersecting if any two members of $\mathcal{F}$ have at least one point in common.

EKR

Theorem

If $\mathcal{F}$ is an intersecting family of k-subsets from a set V of size v, then

EKR

Theorem

If $\mathcal{F}$ is an intersecting family of k-subsets from a set V of size v, then

1. $|\mathcal{F}| \leq \binom{v-1}{k-1}$.

EKR

Theorem

If $\mathcal{F}$ is an intersecting family of k-subsets from a set V of size v, then

1. $|\mathcal{F}| \leq \binom{v-1}{k-1}$.
2. *If equality holds, $\mathcal{F}$ consists of the the k-subsets that contain i, for some i in V.*

Outline

1 Erdős-Ko-Rado
- The Theorem
- Sets to Graphs

2 A Method
- Bounds
- Equality

3 Characterisations
- Kneser
- Derangements

Cocliques

Definition

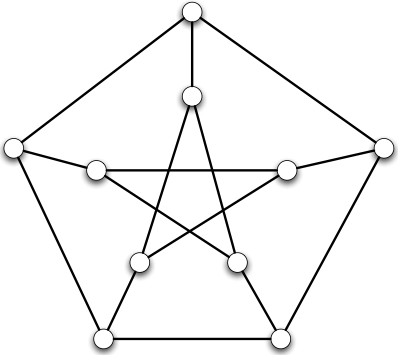
A coclique in a graph is a set of vertices, such that no two vertices in the set are adjacent. The maximum size of a coclique in a graph X is $\alpha(X)$.

A Graph

Definition

The Kneser graph $K_{v:k}$ is the graph with the k-subsets of a v-set as its vertices, where two k-subsets are adjacent if they are disjoint as sets.

$K_{5:2}$



EKR for Graphs

Theorem

We have $\alpha(K_{v:k}) = \binom{v-1}{k-1}$ and a coclique of maximum size consists of the k-subsets that contain i, for some i.

Other Graphs

q-Kneser: The vertices are the k-dimensional subspaces of a vector space of dimension v over $GF(q)$; subspaces are adjacent if their intersection is the zero subspace.

Other Graphs

q-Kneser: The vertices are the k-dimensional subspaces of a vector space of dimension v over $GF(q)$; subspaces are adjacent if their intersection is the zero subspace.

Derangements: The vertices are the permutations of $1, \ldots, n$; two permutations ρ and σ are adjacent if $\rho\sigma^{-1}$ does not have a fixed point.

Other Graphs

q-Kneser: The vertices are the k-dimensional subspaces of a vector space of dimension v over $GF(q)$; subspaces are adjacent if their intersection is the zero subspace.

Derangements: The vertices are the permutations of $1, \dots, n$; two permutations ρ and σ are adjacent if $\rho\sigma^{-1}$ does not have a fixed point.

Partitions: Vertices are the partitions of a set of size n^2 consisting of n cells of size n; two partitions are adjacent if their meet is the discrete partition.

Cocliques

q-Kneser: The subspaces that contain a given 1-dimensional subspace.

Cocliques

q-Kneser: The subspaces that contain a given 1-dimensional subspace.

Derangements: The permutations that map i to j.

Cocliques

q-Kneser: The subspaces that contain a given 1-dimensional subspace.

Derangements: The permutations that map i to j.

Partitions: The partitions with i and j in the same cell.

Outline

1 Erdős-Ko-Rado
- The Theorem
- Sets to Graphs

2 A Method
- Bounds
- Equality

3 Characterisations
- Kneser
- Derangements

Quote

It claims to be fully automatic, but actually you have to push this little button here.
—Gentleman John Killian

A Positive Semidefinite Matrix

Let X be a k-regular graph on v vertices with adjacency matrix A and let τ be the least eigenvalue of A. We define

$$M := (A - \tau I) - \frac{k - \tau}{v} J.$$

Eigenvalues

We have

$$M\mathbf{1} = (k-\tau)\mathbf{1} - \frac{k-\tau}{v} J\mathbf{1} = (k-\tau)\mathbf{1} - (k-\tau)\mathbf{1} = 0.$$

Eigenvalues

We have

$$M\mathbf{1} = (k-\tau)\mathbf{1} - \frac{k-\tau}{v}J\mathbf{1} = (k-\tau)\mathbf{1} - (k-\tau)\mathbf{1} = 0.$$

If $Az = \theta z$ and $\mathbf{1}^T z = 0$, then

$$Mz = (\theta-\tau)z - \frac{k-\tau}{v}Jz = (\theta-\tau)z.$$

So all eigenvalues of M are non-negative and consequently $M \succcurlyeq 0$.

Inequalities

Let S be a coclique in X with characteristic vector x. Then $x^T Ax = 0$ and, since M is positive semidefinite, $x^T Mx \geq 0$. Consequently

$$
\begin{aligned}
0 &\leq x^T Ax - \tau x^T x - \frac{k-\tau}{v} x^T Jx \\
&= 0 - \tau|S| - \frac{k-\tau}{v}|S|^2.
\end{aligned}
$$

Delsarte-Hoffman

Theorem

If X is a k-regular graph on v vertices with least eigenvalue τ, then

$$\alpha(X) \leq \frac{v}{1 - \frac{k}{\tau}}.$$

This is the ratio bound for cocliques, due to Delsarte and Hoffman.

EKR bound

The Kneser graph $K_{v:k}$ has valency

$$\binom{v-k}{k}$$

and least eigenvalue

$$-\binom{v-k-1}{k-1}.$$

EKR1

So

$$\alpha(K_{v:k}) \leq \frac{\binom{v}{k}}{1 + \frac{\binom{v-k}{k}}{\binom{v-k-1}{k-1}}} = \frac{\binom{v}{k}}{1 + \frac{v-k}{k}} = \binom{v-1}{k-1}.$$

q-Kneser

Consider the q-Kneser graph. This has $\begin{bmatrix} v \\ k \end{bmatrix}$ vertices, valency

$$q^{k^2} \begin{bmatrix} v - k \\ k \end{bmatrix}$$

and its least eigenvalue is

$$-q^{k(k-1)} \begin{bmatrix} v - k - 1 \\ k - 1 \end{bmatrix}$$

EKR2

The ratio bound is

$$\begin{bmatrix} v-1 \\ k-1 \end{bmatrix}$$

which is realized by the k-subspaces that contain a given 1-dimensional subspace.

The Derangement Graph

The vertices of the derangement graph $D(n)$ are the permutations of $1, \dots, n$; two permutations ρ and σ are adjacent if $\rho\sigma^{-1}$ does not have a fixed point.

The Derangement Graph

The vertices of the derangement graph $D(n)$ are the permutations of $1, \ldots, n$; two permutations ρ and σ are adjacent if $\rho\sigma^{-1}$ does not have a fixed point.

cocliques: The set $S_{i,j}$ of permutations that map i to j is a coclique, of size $(n-1)!$.

The Derangement Graph

The vertices of the derangement graph $D(n)$ are the permutations of $1, \ldots, n$; two permutations ρ and σ are adjacent if $\rho\sigma^{-1}$ does not have a fixed point.

cocliques: The set $S_{i,j}$ of permutations that map i to j is a coclique, of size $(n-1)!$.

cliques: Latin squares are cliques. In particular, if G is a regular subgroup of $\mathrm{Sym}(n)$, then the elements of G form a clique of size n, as do its cosets. (This implies that $\alpha(D(n)) = (n-1)!$.)

Eigenvalues of $D(n)$

???

Outline

1 Erdős-Ko-Rado
- The Theorem
- Sets to Graphs

2 A Method
- Bounds
- Equality

3 Characterisations
- Kneser
- Derangements

Remarks

1. If S is a coclique with characteristic vector x and $|S| = v/(1 - \frac{k}{\tau})$ then $x^T M x = 0$.

Remarks

1. If S is a coclique with characteristic vector x and $|S| = v/(1 - \frac{k}{\tau})$ then $x^T Mx = 0$.
2. If $M \succcurlyeq 0$ and $x^T Mx = 0$, then $Mx = 0$. (Proof: $M = U^T U$ and $x^T U^T Ux = 0$ if and only if $Ux = 0$.)

Remarks

1. If S is a coclique with characteristic vector x and $|S| = v/(1 - \frac{k}{\tau})$ then $x^T M x = 0$.
2. If $M \succcurlyeq 0$ and $x^T M x = 0$, then $Mx = 0$. (Proof: $M = U^T U$ and $x^T U^T U x = 0$ if and only if $Ux = 0$.)
3. Hence if $y := x - \frac{|S|}{v}\mathbf{1}$, then $My = 0$.

Eigenvectors

Theorem

If X is a k-regular graph on v vertices with least eigenvalue τ and x is the characteristic vector of a coclique with size $v/(1-\frac{k}{\tau})$, then $x - \frac{|S|}{v}\mathbf{1}$ is an eigenvector for $A(X)$, with eigenvalue τ.

Eigenvectors

Theorem

If X is a k-regular graph on v vertices with least eigenvalue τ and x is the characteristic vector of a coclique with size $v/(1-\frac{k}{\tau})$, then $x - \frac{|S|}{v}\mathbf{1}$ is an eigenvector for $A(X)$, with eigenvalue τ.

Proof.

If $y := x - \frac{|S|}{v}\mathbf{1}$, then

$$0 = My = (A - \tau I)y - \frac{k-\tau}{v}Jy = (A - \tau I)y.$$

□

Eigenspaces

Let W be the $\binom{v}{k} \times v$ matrix whose rows are the characteristic vectors of the k-subsets of $\{1, \ldots, v\}$. Then the eigenspace for the least eigenvalue of $K_{v:k}$ consists of the vectors in $\operatorname{col}(W)$ that are orthogonal to $\mathbf{1}$.

Eigenspaces

Let W be the $\binom{v}{k} \times v$ matrix whose rows are the characteristic vectors of the k-subsets of $\{1, \dots, v\}$. Then the eigenspace for the least eigenvalue of $K_{v:k}$ consists of the vectors in $\operatorname{col}(W)$ that are orthogonal to $\mathbf{1}$.

Corollary

If x is the characteristic vector of a coclique of size $\binom{v-1}{k-1}$ in $K_{v:k}$, then $x \in \operatorname{col}(W)$.

Our Problem

Prove that if Wh is a 01-vector, then $h = e_i$ for some i.

Outline

1 Erdős-Ko-Rado
- The Theorem
- Sets to Graphs

2 A Method
- Bounds
- Equality

3 Characterisations
- Kneser
- Derangements

A Polytope

Suppose x is the characteristic vector of a coclique S in $K_{v:k}$ with size $\binom{v-1}{k-1}$. Then

$$x = Wh$$

for some h. If we view the rows of W as points in $\mathbb{R}^v$, they form the vertices of a convex polytope and the support of x is a face.

Faces

Theorem

A face of the polytope generated by the rows of W consists of the k-subsets that contain a given subset S, and are contained in a given subset T. All faces arise in this way.

Proving EKR, I

By the theorem, the support of Wh consists of the k-subsets α such that

$$S \subseteq \alpha \subseteq T$$

for some sets S and T. If $S \neq \emptyset$, we are done.

Proving EKR, II

So assume $S = \emptyset$. Then the support of Wh consists of all k-subsets of T. Since the support is an intersecting family, $|T| \le 2k-1$ and consequently our family has size

$$\binom{2k-1}{k} = \binom{2k-1}{k-1}.$$

But $v \ge 2k+1$ and

$$\binom{2k-1}{k-1} < \binom{2k}{k-1} = \binom{v-1}{k-1}.$$

Thus if $S = \emptyset$, the support of Wh is not an intersecting family of maximal size.

A Second Proof

If $\alpha \in S$ and β is a k-subset disjoint from S, then $(Wh)_\beta = 0$. Let N be the submatrix formed by the rows of W that are indexed by subsets in the complement of α. Then

$$Nh = 0$$

A Second Proof

If $\alpha \in S$ and β is a k-subset disjoint from S, then $(Wh)_\beta = 0$. Let N be the submatrix formed by the rows of W that are indexed by subsets in the complement of α. Then

$$Nh = 0$$

Further we can write N in the form

$$N = \begin{pmatrix} 0 & N_1 \end{pmatrix}$$

where the initial zero columns are indexed by the elements of α.

Rank

The rows of N_1 are indexed by the k-subsets disjoint from α, the columns by the $v-k$ points not in α. So N_1 is $W_{v-k,k}$ and, if $v-k>k$, its columns are linearly independent.

Rank

The rows of N_1 are indexed by the k-subsets disjoint from α, the columns by the $v-k$ points not in α. So N_1 is $W_{v-k,k}$ and, if $v-k>k$, its columns are linearly independent.

Hence if

$$\begin{pmatrix} 0 & N_1 \end{pmatrix} h = 0,$$

then α contains the support of h.

Outline

1 Erdős-Ko-Rado
- The Theorem
- Sets to Graphs

2 A Method
- Bounds
- Equality

3 Characterisations
- Kneser
- Derangements

Babbage to Tennyson

Sir, in your otherwise beautiful poem (The Vision of Sin) there is a verse which reads:

Every moment dies a man,
every moment one is born.

Obviously this cannot be true and I suggest that in the next edition you have it read:

Every moment dies a man,
every moment one-and-one-sixteenth is born.

Even this value is slightly in error but should be sufficiently accurate for the purposes of poetry.

The Bound

The cosets of a regular subgroup of $\mathrm{Sym}(n)$ form a partition of the vertices into $(n-1)!$ cliques of size n, and since any coclique contains at most one vertex from each clique, it follows that

$$\alpha(D(n)) \leq (n-1)!.$$

Clique-Coclique

Theorem

If the graph X has a set of cliques of the same size that cover each vertex the same number of times, then $\alpha(X)\omega(X) \leq |V(X)|$.

Proof

$\mathcal{C}$: the set of cliques in our clique cover

Proof

$\mathcal{C}$: the set of cliques in our clique cover

N: the incidence matrix for the vertices versus cliques.

Proof

$\mathcal{C}$: the set of cliques in our clique cover

N: the incidence matrix for the vertices versus cliques.

ν: the number of cliques per vertex (so $N\mathbf{1} = \nu\mathbf{1}$).

Proof

$\mathcal{C}$: the set of cliques in our clique cover

N: the incidence matrix for the vertices versus cliques.

ν: the number of cliques per vertex (so $N\mathbf{1} = \nu\mathbf{1}$).

Proof

$\mathcal{C}$: the set of cliques in our clique cover

N: the incidence matrix for the vertices versus cliques.

ν: the number of cliques per vertex (so $N\mathbf{1} = \nu\mathbf{1}$).

If x is the characteristic vector of a coclique, then $x^T N \leq \mathbf{1}^T$ and hence $x^T N\mathbf{1} \leq |\mathcal{C}|$. On the other hand

$$x^T N\mathbf{1} = \nu x^T \mathbf{1} = \nu|S|$$

and so $|S| \leq |\mathcal{C}|/\nu$. Since $|\mathcal{C}|\omega(X) \geq \nu|V(X)|$, the result follows.

Equality

Suppose we have a uniform clique cover and equality holds in the clique-coclique bound. Let S and C respectively be a coclique and clique of maximum size with characteristic vectors x_S and x_T. Then $|S \cap C| = 1$ and the vectors

$$x_S - \frac{|S|}{|V(X)|}\mathbf{1}, \qquad x_C - \frac{|C|}{|V(X)|}\mathbf{1}$$

are orthogonal.

Bad News

- Each eigenspace of the derangement graph is a sum of irreducible modules for $\mathrm{Sym}(n)$. These modules are indexed by integer partitions and each occurs exactly once.

Bad News

- Each eigenspace of the derangement graph is a sum of irreducible modules for $\mathrm{Sym}(n)$. These modules are indexed by integer partitions and each occurs exactly once.
- There are algorithms to compute the eigenvalue associated to the module, but they do not allow us to read off the least eigenvalue, and so we cannot use the ratio bound

Bad News

- Each eigenspace of the derangement graph is a sum of irreducible modules for $\mathrm{Sym}(n)$. These modules are indexed by integer partitions and each occurs exactly once.
- There are algorithms to compute the eigenvalue associated to the module, but they do not allow us to read off the least eigenvalue, and so we cannot use the ratio bound

Bad News

- Each eigenspace of the derangement graph is a sum of irreducible modules for $\mathrm{Sym}(n)$. These modules are indexed by integer partitions and each occurs exactly once.
- There are algorithms to compute the eigenvalue associated to the module, but they do not allow us to read off the least eigenvalue, and so we cannot use the ratio bound :-(

Good News

- We can find cliques whose characteristic vectors span the orthogonal complement of the module associated to the partition $(n-1,1)$.

Good News

- We can find cliques whose characteristic vectors span the orthogonal complement of the module associated to the partition $(n-1,1)$.
- By the clique-coclique bound, this implies that the characteristic vector of a coclique of size $(n-1)!$ must lie in the module associated to the partition $(n-1,1)$.

Good News

- We can find cliques whose characteristic vectors span the orthogonal complement of the module associated to the partition $(n-1,1)$.
- By the clique-coclique bound, this implies that the characteristic vector of a coclique of size $(n-1)!$ must lie in the module associated to the partition $(n-1,1)$.
- From this it follows ($\mathrm{rk}(N_1)$ argument, or use perfect matching polytope) that a coclique of size $(n-1)!$ must be one of the sets $S_{i,j}$.

Problems

1. Perfect matchings in K_{2m}.

Problems

1. Perfect matchings in K_{2m}.
2. Partitions graphs with $n \geq 4$.