Probabilistic Graph and Hypergraph Matching

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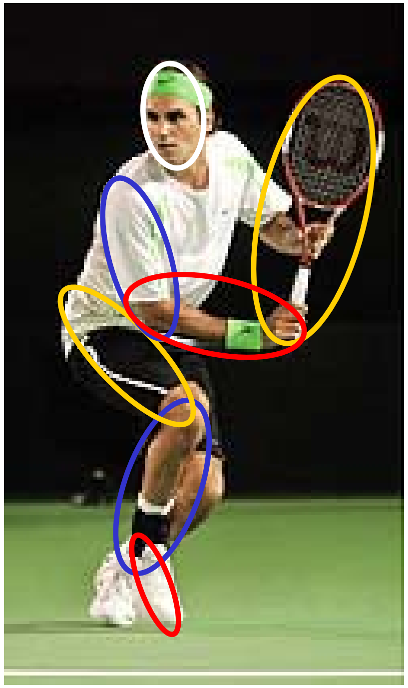
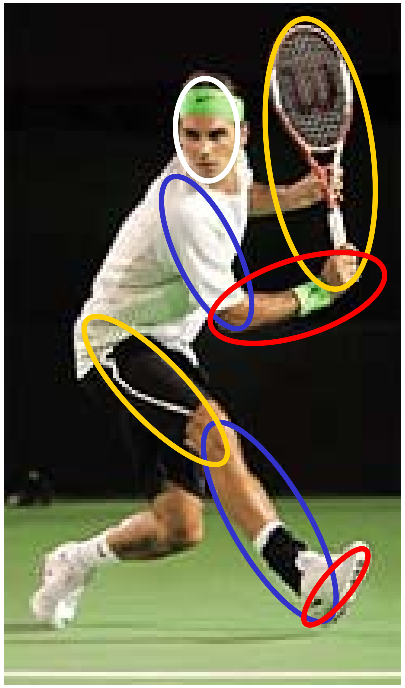
Example: Object Matching

- No global affine transform.



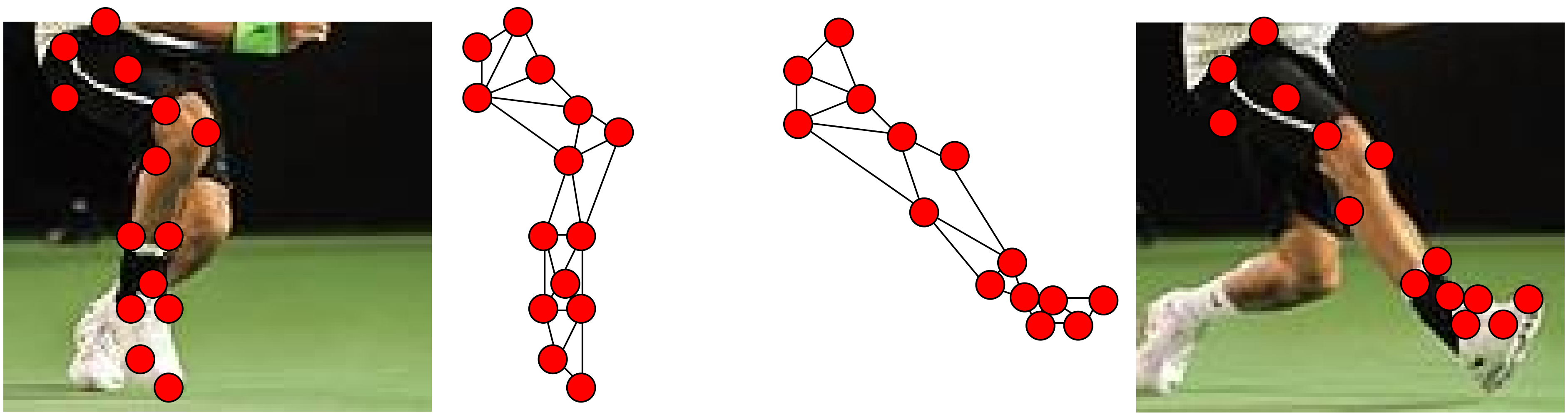


- Local affine transforms + small non-rigid motion.
- Match by local features + local structure.

Images from: www.operationdoubles.com/one_handed_backhand_tennis.htm

Hypergraph Matching in Computer Vision

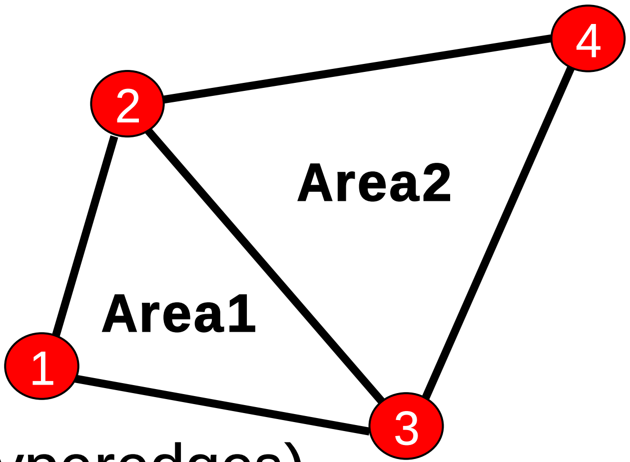
- In **graph matching**, we describe objects as graphs (features → nodes, distances → edges) and match objects by matching graphs.



- **<u>Problem:</u>** Distances are not affine invariant.

Hypergraph Matching in Computer Vision

- Affine invariant properties.
  - Properties of four or more points.
  - Example: area ratio, Area1 / Area2
  - Describe objects as hypergraphs (features → nodes, area ratio → hyperedges)
  - Match objects by doing **Hypergraph Matching**.



- In general, if *n* points are required to solve the local transformation, $d = n+1$ points are required for an invariant property.

Related Work
Hypergraph matching

- Hypergraph matching:
  - Wong, Lu & Rioux, PAMI 1989
  - Sabata & Aggarwal, CVIU 1996
  - Demko, GbR 1998
  - Bunke, Dickinson & Kraetzl, ICIAP 2005
- All search for an **exact** matching.
  - Edges are matched to edges of the **exact same label**.
  - **Search algorithms** for the largest sub-isomorphism.

Unrealistic

- We are interested in an **inexact** matching.
  - Edges are matched to edges with similar labels.
  - Find the best matching according to some score function.

Related Work
Inexact **Graph** Matching

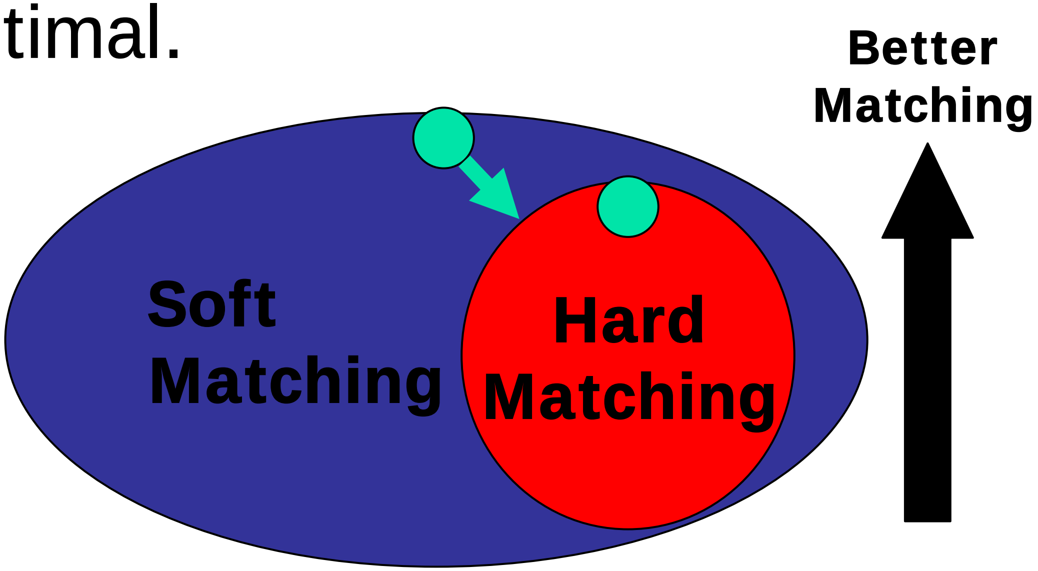
- Popular line of works: Continuous relaxation
  - As an SDP problem
    - Schellewald & Schnörr, CVPR 2005
  - As a spectral decomposition problem
    - Leordeanu & Hebert, ICCV 2005; Cour, Srinivasan & Shi, NIPS 2006
  - Iterative Linear approximations using Taylor expansions
    - Gold & Rangarajan, CVPR 1995
  - And many more.
- Some continuous relaxation may be interpreted as **soft matching**.

<u>Our work differ:</u>

We assume probabilistic interpretation of the input and extract probabilistic matching

From Soft to Hard

- Given the optimal soft solution X, the nearest hard matching is found by solving a Linear Assignment Problem.
  - The two steps (soft matching and nearest hard matching) are optimal.
  - The overall hard matching is not optimal (NP-hard).



Hypergraph Matching

- Two *directed* hypergraphs of degree d, $G=(V,E)$ and $G'=(V',E')$.
  - A hyper-edge is an ordered d-tuple of vertices.
  - Include the *undirected* version as a private case.
- <u>Matching:</u> $m : V \rightarrow V'$
- Induce edge matching, $m : E \rightarrow E'$,

$$m(v_1,\dots,v_d) = (m(v_1),\dots,m(v_d))$$

Probabilistic Hypergraph Matching

- Input: Probability that an edge from $e \in E$ match to an edge in $e' \in E'$:

$$S_{e,e'} = \Pr\left(m(e) = e' \mid G, G'\right)$$

- Output: Probability that two vertices match

$$X_{v,v'} = \Pr\left(m(v) = v' \mid G, G'\right)$$

- We will derive an algebraic connection between *S* and *X*, and then use it for finding the optimal *X*.

Kronecker Product

- Kronecker product between an *i*x*j* matrix A to a *k*x*l* matrix B is a *ik*x*jl* matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1i}B \\ \vdots & \ddots & \vdots \\ a_{j1}B & \cdots & a_{ij}B \end{bmatrix}$$

$$\otimes_{i=1}^{d} A_i = A_1 \otimes \cdots \otimes A_d$$

$$\otimes^{d} A = \otimes_{i=1}^{d} A$$

$S \leftrightarrow X$ connection

- <u>Assumption</u> $m(v_1) \perp m(v_2) \mid G, G'$
- <u>Proposition ($S \leftrightarrow X$ connection)</u>

$$\boxed{S = \otimes^d X}$$

- <u>Proof</u>

$$S_{e,e'} = \Pr\big(m(e) = e' \mid G, G'\big) =$$

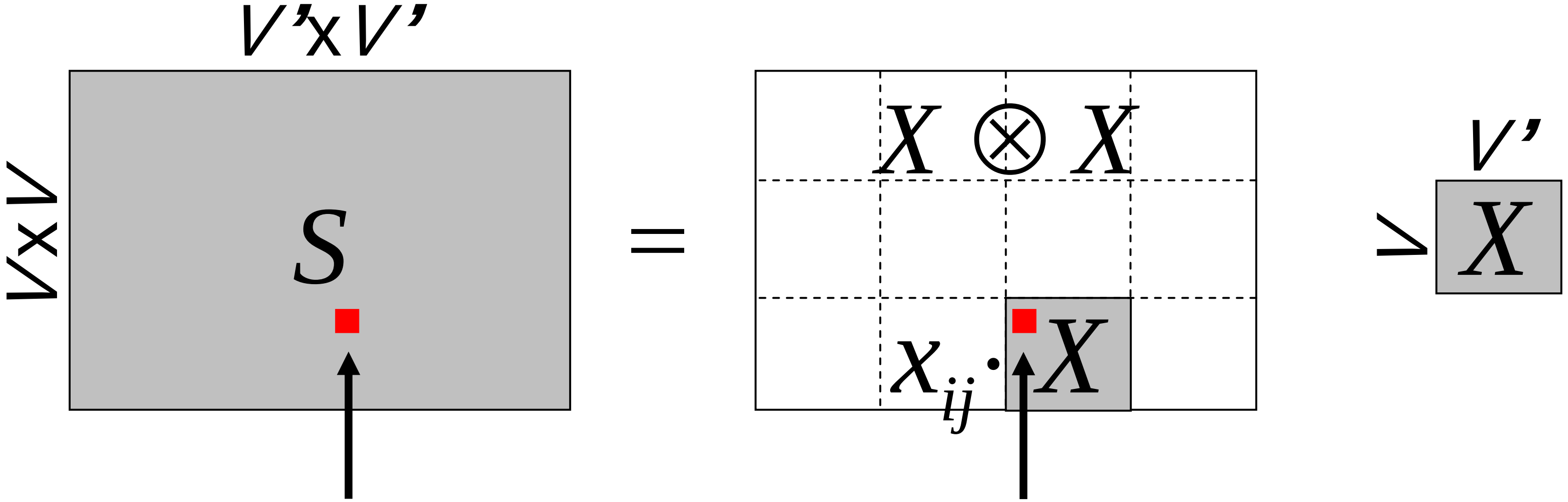
$$= \prod_{i=1}^{d} \Pr\big(m(v_i) = v_i' \mid G, G'\big) =$$

$$= \prod_{i=1}^{d} X_{v_i, v_i'}$$

$$e = (v_1, \cdots, v_d),$$
$$e' = (v_1', \cdots, v_d')$$

$S \leftrightarrow X$ connection for graphs

$$S = X \otimes X$$



$$S_{(v_1,v_2),(v_1',v_2')} = X_{v_1,v_1'} \bullet X_{v_2,v_2'}$$

Globally Optimal Soft Hypergraph Matching

- Nearest $\otimes^d X$ to *S*, where *X* is a valid matrix of probabilities:

$$\min_X \quad dist\left(S, \otimes^d X\right)$$
$$s.t. \qquad X \geq 0,\ X\mathbf{1} \leq \mathbf{1},\ X^T\mathbf{1} \leq \mathbf{1}$$

- $X\mathbf{1} \leq \mathbf{1},\ X^T\mathbf{1} \leq \mathbf{1}$ ➔
  - Vertex can be left unmatched.
  - With equalities, all vertices must be matched.

Cour, Srinivasan & Shi 2006

- Our result can explain some previously used heuristics.
- <u>Cour et al 2006 preprocessing:</u> Replace *S* with the nearest doubly stochastic matrix (in relative entropy) before any other graph matching algorithm.
- <u>Proposition:</u> For $X \geq 0$, X is doubly stochastic iff $\otimes^d X$ is doubly stochastic.
- ➔ $S = \otimes^d X$ is doubly stochastic.

Globally Optimal Soft Hypergraph Matching

$$\min_{X} \quad dist\left(S, \otimes^{d} X\right)$$

$$s.t. \quad X \geq 0, X\mathbf{1} \leq \mathbf{1}, X^{T}\mathbf{1} \leq \mathbf{1}$$

- We use the Relative Entropy (Maximum Likelihood) error measure,

$$dist(A,B) = D(A \| B) = \sum_{i,j} A_{ij} \log\left(\frac{A_{ij}}{B_{ij}}\right) - A_{ij} + B_{ij}$$

- **Global Optimum, Efficient.**

Globally Optimal Soft Hypergraph Matching

$$X^* = \arg\min_X D\left(S \,\|\, \otimes^d X\right)$$

<u>Define:</u>

$$Y_{v,v'} = \sum_{i=1}^{d} \sum_{\substack{e \mid e_i = v \\ e' \mid e'_i = v'}} S_{e,e'}$$

$$X^* = \arg\min_X \quad D\left(Y \,\|\, X\right) + \left(\mathbf{1}^T X \mathbf{1}\right)^d - \left(\mathbf{1}^T X \mathbf{1}\right)$$

Convex problem, with **| *V* | x | *V'* |** inputs and outputs!

Globally Optimal Soft Hypergraph Matching

$$X^* = \arg\min_X \quad D(Y \| X) + \left(\mathbf{1}^T X \mathbf{1}\right)^d - \left(\mathbf{1}^T X \mathbf{1}\right)$$

Define $k = \mathbf{1}^T X \mathbf{1}$, the number of matches.

$$X^*(k) = \arg\min_X \quad D(Y \| X) \quad s.t.$$

$$X \geq 0,\ X\mathbf{1} \leq \mathbf{1},\ X^T\mathbf{1} \leq \mathbf{1},\ \mathbf{1}^T X \mathbf{1} = k$$

- $X^*(k)$ is convex in k.
- We give optimal solution for $X^*(k)$, and solve for k numerically (convex minimization in single variable).

Globally Optimal Soft Hypergraph Matching

$$\min_X \quad D\left(Y \,\|\, X\right) \quad s.t. \quad X \in C_1 \cap C_2 \cap C_3$$

$$C_1 = \left\{ X \mid X \geq 0,\ X\mathbf{1} \leq \mathbf{1} \right\}$$

$$C_2 = \left\{ X \mid X \geq 0,\ X^T\mathbf{1} \leq \mathbf{1} \right\}$$

$$C_3 = \left\{ X \mid X \geq 0, \mathbf{1}^T X\mathbf{1} = k \right\}$$

- Define three sub-problems (j=1,2,3):

$$P_j(H) = \min_{X \in C_j} \quad D\left(Y \,\|\, X\right) - trace\left(X^T H\right)$$

- Each has an optimal closed form solution.

Successive Projections
[Tseng 93, Censor & Reich 98]

$$X_0^{(t)} = X_3^{(t-1)}, \quad f(X) = D(Y \| X)$$

- Set $\lambda_j^{(0)} = 0, \quad X_j^{(0)} = Y.$
- For t=1,2,... till convergence:
  - For j = 1,2,3:
    - $X_j^{(t)} = P_j\left(\lambda_j^{(t-1)} + \nabla f\left(X_{j-1}^{(t)}\right)\right)$
    - $\lambda_j^{(t)} = \lambda_j^{(t-1)} + \nabla f\left(X_{j-1}^{(t)}\right) - \nabla f\left(X_j^{(t)}\right)$

Optimal!

Globally Optimal Soft Hypergraph Matching

- When the hypergraphs are of the same size, and all vertices has to be matched,

$$X\mathbf{1} = X^T\mathbf{1} = \mathbf{1},$$

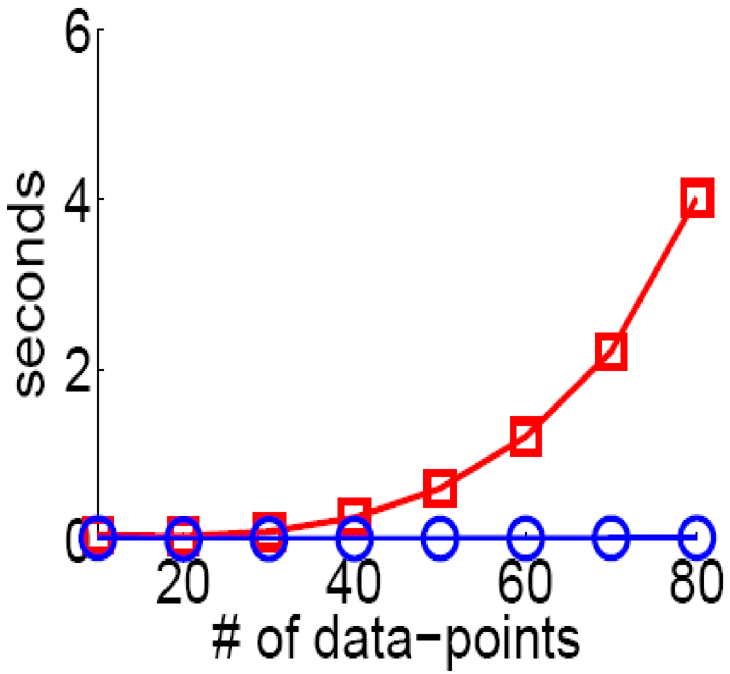
our algorithm reduces to the Sinkhorn algorithm for nearest doubly stochastic matrix in relative entropy.

Sampling

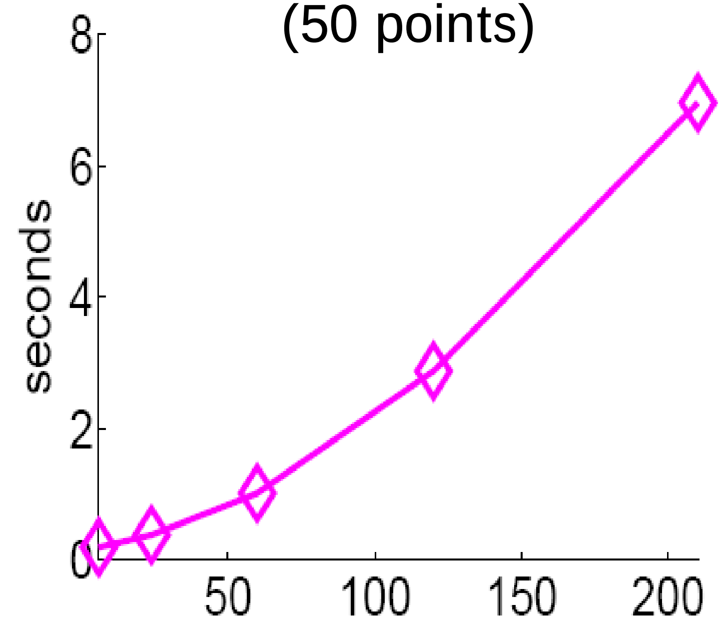
- Given *Y*, the problem size reduce to $|V| \times |V'|$.
- <u>Calculate *Y*:</u> simple sum on all hyper-edges.
- <u>Problem:</u> Compute *S*, the hyper-edge to hyper-edge correlation.
- <u>Sampling heuristic:</u> For each vertex, use only *z* closest hyper-edges.
- Heuristic applies to transformation that are locally affine (but globally not affine).
- $O(|V| \cdot |V'| \cdot z^2)$ correlations.

Runtime

Without edge correlations time



With hyperedge correlations time (50 points)



Our scheme (graphs)

Spectral Matching Leordeanu05

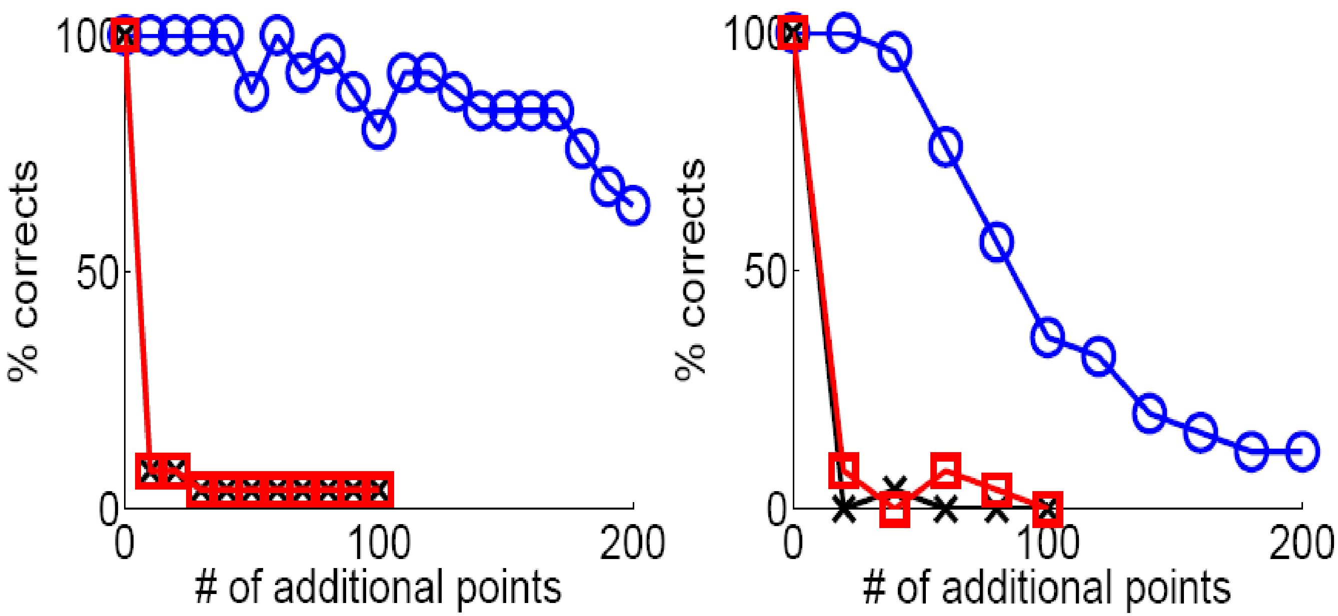
Our scheme (hypergraphs)

Experiments on **Graphs**

- Two duplicates of 25 points.
- Graphs based on distances.
- Additional random points.

to a single graph

to both graphs

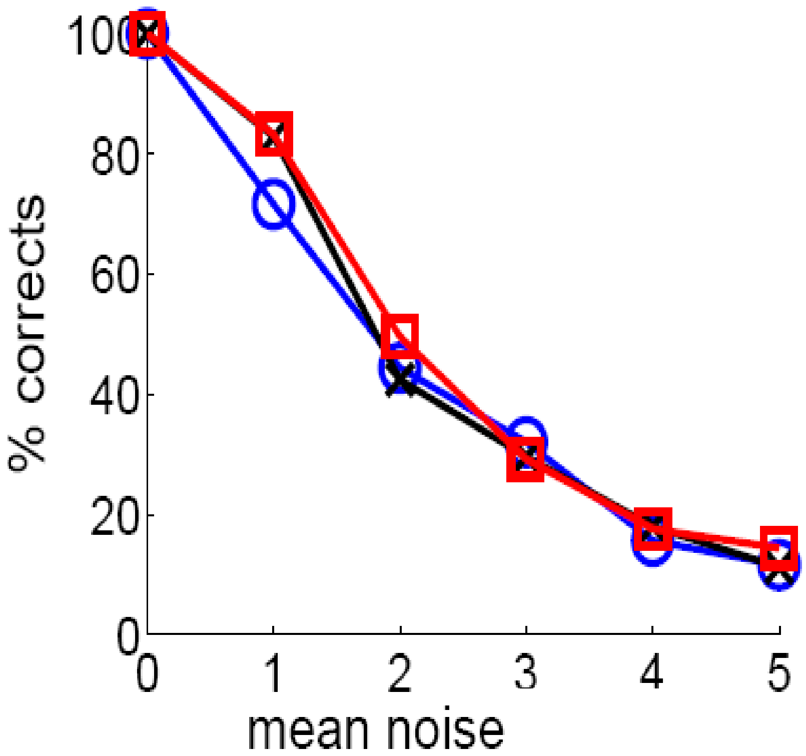


Our scheme

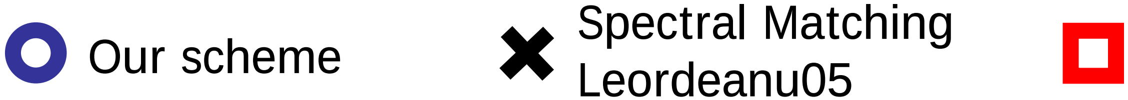
Spectral Matching Leordeanu05

Spectral Matching Leordeanu05 with Cour06 preprocessing

Experiments on **Graphs**



- Mean distance between neighboring points is 1.
- One duplicate distorted with a random noise.
- Spectral uses Frobenius norm – should have better resilience to additive noise.
- Due to the global optimal solution, Relative Entropy shows comparable results.



Limitations of Graphs

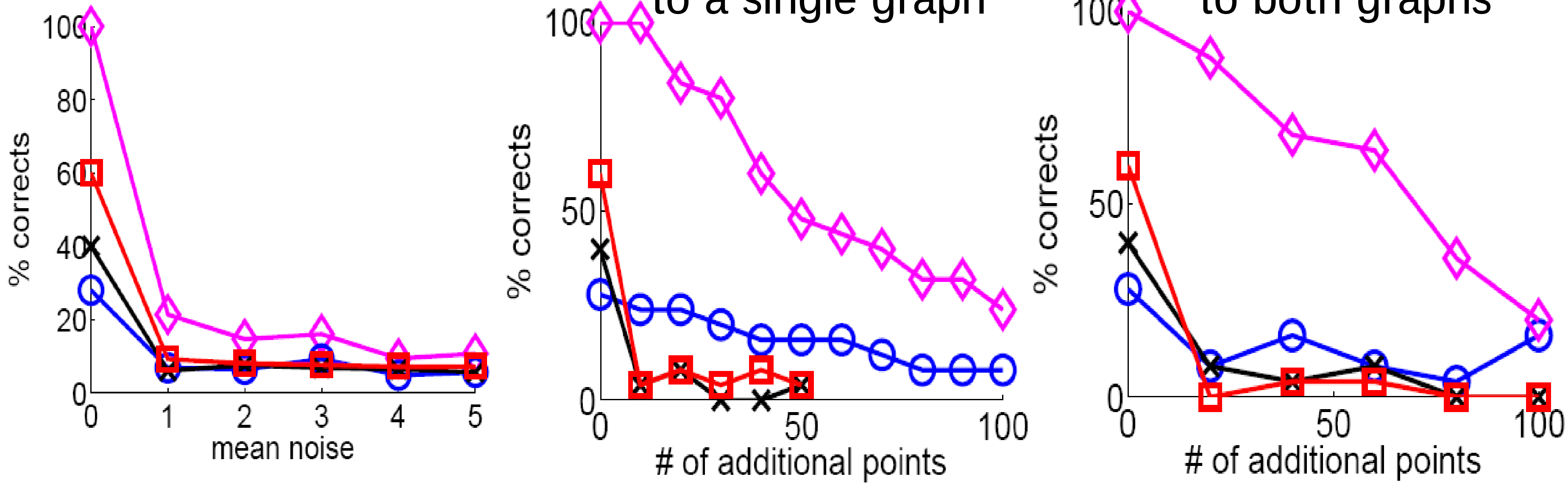
Affine Transformation (doesn’t preserve distances)

random distortion

additional points

to a single graph

to both graphs



Our scheme (graphs)

Our scheme (hypergraphs, z=60)

Spectral Matching Leordeanu05

Spectral Matching Leordeanu05 with Cour06 preprocessing

Feature Matching in Computer Vision

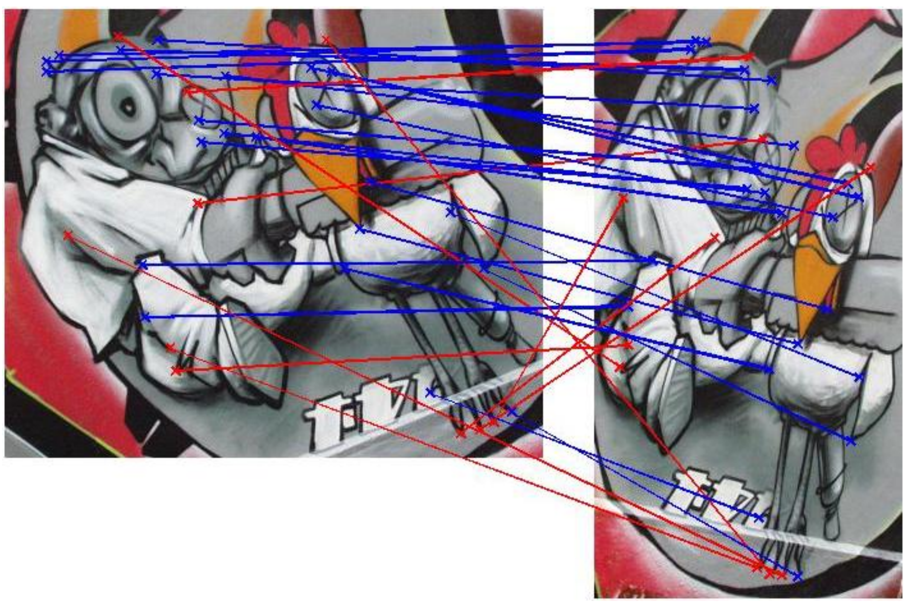
- Describe objects by local features (e.g., SIFT).
- Match objects by matching features.

- Based solely on local appearance:
  - Different features might look the same.
  - Same feature might look differently.

Global Affine Transformation

Spectral Graph Matching
based on distances

10/33 mismatches



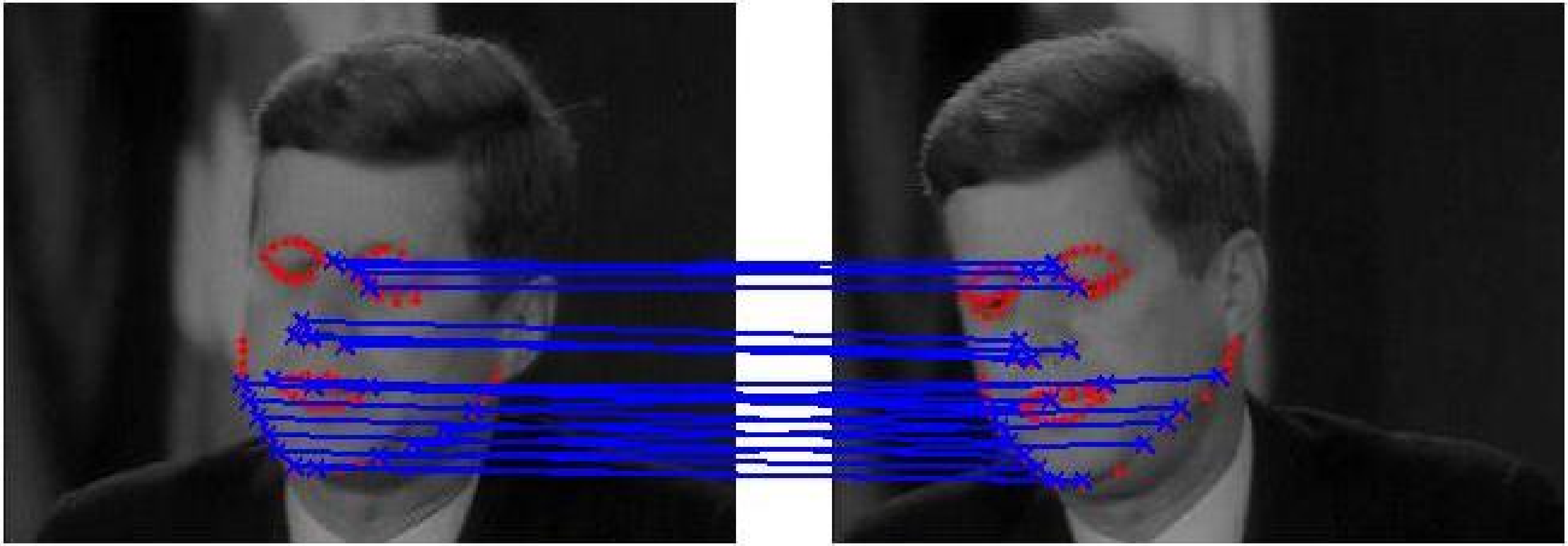
Hypergraph Matching
based on area ratio

no mismatches

Images from: www.robots.ox.ac.uk/vgg/research/affine/index.html

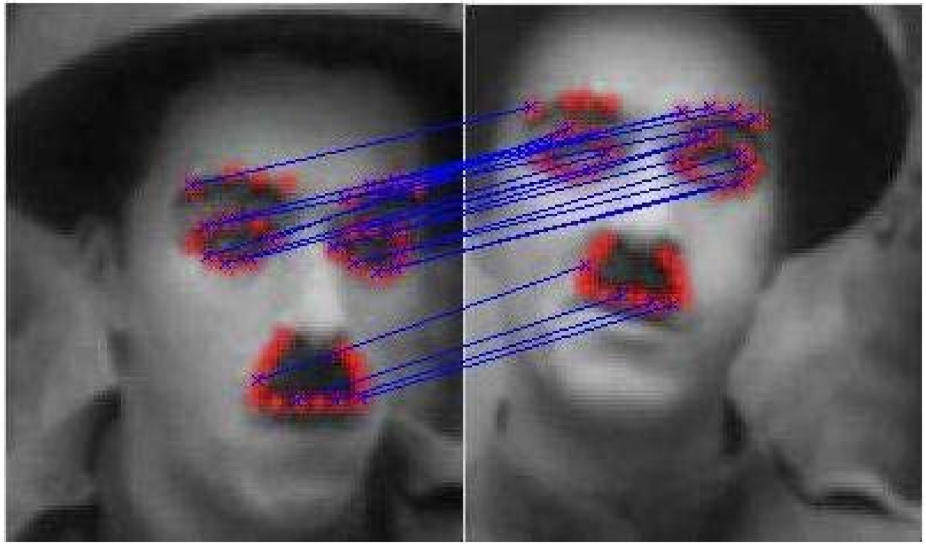
Non-rigid Matching

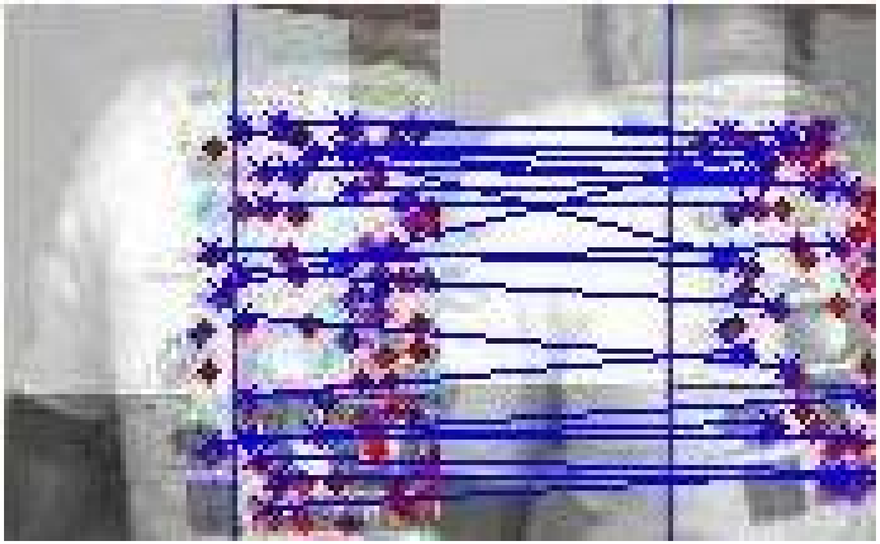
- Match first and last frames of a 200 frames video (6 seconds), using [Torresani & Bregler, Space-Time Tracking, 2002] features.



Videos and points from: movement.stanford.edu/nonrig/

Non-rigid Matching





Videos and points from: movement.stanford.edu/nonrig/

Summary

- **Structure translates to hypergraphs**, not graphs.
- Probabilistic interpretation leads to a simple connection between input and output:

$$S = \otimes^d X$$

- **Globally Optimal** solution under Relative Entropy (Maximum Likelihood).
- **Efficient** for both graphs and hypergraphs.
- Apply to graph matching problems as well.