Path integral control from probabilistic viewpoint

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SNN Adaptive Intelligence

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Motivation

Path integral control

Dynamics

$$dX(t) = [b(X(t)) + \sigma(X(t))v(t)]\ dt + \sigma(X(t))\ dW(t).$$

Cost

$$\mathbb{E}^x \left[m(X_T) + \int_t^T c(X(s))\ ds + \frac{1}{2\beta} \int_t^T v^2(s, X(s))\ ds \right].$$

After manipulation using PDEs, we obtain an optimal value of the form

$$-\frac{1}{\beta} \ln \mathbb{E}^{t,x} \exp \left(-\beta \left(m(X_T) + \int_t^T c(X(s)) \right) \right).$$

Understanding path integral control

- Is the PDE approach necessary?
- Can we generalize?

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- Can we generalize?

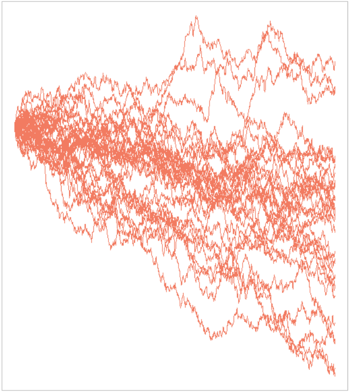
Kullback-Leibler divergence / relative entropy

- Ω probability space, $\mathbb{P}$, $\mathbb{Q}$ probability measures;
- $\mathrm{KL}(\mathbb{P}||\mathbb{Q}) := \int_\Omega \ln\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right) d\mathbb{P}$.
- $\mathrm{KL}(\mathbb{P}||\mathbb{Q}) \geq 0$ for all $\mathbb{P}$.
- $\mathrm{KL}(\mathbb{P}||\mathbb{Q}) = 0$ iff $\mathbb{P} = \mathbb{Q}$.
- $\mathrm{KL}(\mathbb{P}||\mathbb{Q}) \neq \mathrm{KL}(\mathbb{Q}||\mathbb{P})$, in general.
- $\mathrm{KL}(\mathbb{P}||\mathbb{Q}) = \infty$ if $\mathbb{P}$ is not absolutely continuous with respect to $\mathbb{Q}$, i.e. if $\mathbb{P}(A) > 0$ for some A with $\mathbb{Q}(A) = 0$.

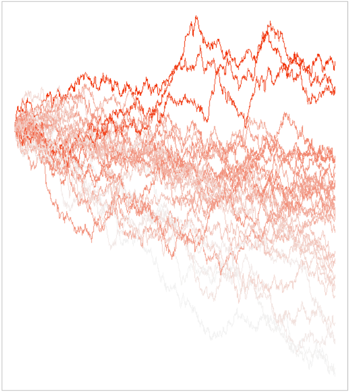
Kullback-Leibler divergence weighted optimization

- Ω probability space, probability measure $\mathbb{Q}$.
- C (*cost*) random variable.
- Goal: find probability measure $\mathbb{P}$ that
  (a) minimizes expected cost,
  (b) while remaining close to $\mathbb{Q}$ (and in particular absolutely continuous with respect to $\mathbb{Q}$).
- minimize $\mathbb{E}^{\mathbb{P}} C + \frac{1}{\beta} \mathrm{KL}(\mathbb{P}||\mathbb{Q}) = \int_\Omega C + \frac{1}{\beta} \ln \left(\frac{d\mathbb{P}}{d\mathbb{Q}} \right) \, d\mathbb{P}$ with respect to $\mathbb{P}$.
- Minimization of $\mathbb{E}^{\mathbb{P}} C + \frac{1}{\beta} \mathrm{KL}(\mathbb{P}||\mathbb{Q})$ over probability measures $\mathbb{P}$ yields
$$\mathbb{P}(A) = \frac{\int_A \exp(-\beta C) \, d\mathbb{Q}}{Z}.$$
- Optimal value: $-\frac{1}{\beta} \ln \mathbb{E}^{\mathbb{Q}} \exp(-\beta C)$.

Girsanov's theorem I



30 paths of a Brownian motion with negative drift



The same paths reweighted according to the Girsanov formula

$$\exp\left(\int_0^T v(s)\, dW(s) - \frac{1}{2}\int_0^T v(s)^2\, ds\right)$$

Girsanov's theorem II

First an example

Suppose $\mathbb{P} \sim N(\mu, 1)$ on $\mathbb{R}$ and $\mathbb{Q} \sim N(0, 1)$. Then $d\mathbb{Q}(\omega) = \frac{1}{\sqrt{2\pi}} \exp\left(-\omega^2/2\right)$, $d\mathbb{P}(\omega) = \frac{1}{\sqrt{2\pi}} \exp\left(-(\omega-\mu)^2/2\right)$. Hence

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{\exp\left(-(\omega-\mu)^2/2\right)}{\exp\left(-\omega^2/2\right)} = \exp\left(\mu\omega - \tfrac{1}{2}\mu^2\right).$$

Girsanov theorem

- $\Omega = C[0, T]$ with $\mathbb{Q}$ Wiener measure: $\omega \in \Omega$ represent paths of a Brownian motion.
- v stochastic process.

Define $Z = \exp\left(\int_0^T v(s)\, dW(s) - \frac{1}{2}\int_0^T v(s)^2\, ds\right)$.
Then

- $\mathbb{P} = Z\mathbb{Q}$ is a probability measure on Ω
- Under this measure, $t \mapsto \omega(t) - \int_0^t v(s)\, ds$ is a Brownian motion.

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KL weighted optimization of diffusions I

- $(W(t))_{t\geq 0}$ Brownian motion.
- Diffusion X defined by

$$dX(t) = b(X(t))\,dt + \sigma(X(t))\,dW(t), \quad X(0) = x \in \mathbb{R},$$

- Girsanov: $t \mapsto W^v(t) := W(t) - \int_0^t v(s)\,ds$ is a Brownian motion under measure $\mathbb{P}^v$ defined by

$$\frac{d\mathbb{P}^v}{d\mathbb{Q}}(\omega) = \exp\left(\int_0^T v(s,\omega)\,dW(s,\omega) - \frac{1}{2}\int_0^T v^2(s,\omega)\,ds\right).$$

- X satisfies

$$dX(t) = [b(X(t)) + \sigma(X(t))v(t)]\,dt + \sigma(X(t))\,dW^v(t).$$

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KL divergence for controlled diffusion

Girsanov density

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$$\begin{aligned}
\mathrm{KL}(\mathbb{P}^v||\mathbb{Q}) &= \mathbb{E}^v \ln\left(\frac{d\mathbb{P}^v}{d\mathbb{P}}\right)\\
&= \mathbb{E}^v\left[\int_0^T v(s,X(s))\, dW(s) - \frac{1}{2}\int_0^T v^2(s,X(s))\, ds\right]\\
&= \mathbb{E}^v\left[\int_0^T v(s,X(s))\, dW^v(s) + \frac{1}{2}\int_0^T v^2(s,X(s))\, ds\right]\\
&= \mathbb{E}^v\left[\frac{1}{2}\int_0^T v^2(s,X(s))\, ds\right].
\end{aligned}$$

So $C = m(X_T) + \int_0^T c(X(s))\, ds$ gives 'familiar'

$$\mathbb{E}^v C + \frac{1}{\beta}\mathrm{KL}(\mathbb{P}^v||\mathbb{Q}) = \mathbb{E}^v m(X_T) + \int_0^T c(X(s))\, ds + \frac{1}{2\beta}\int_0^T v^2(s,X(s))\, ds.$$

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KL weighted optimization of diffusions II - classical solution

$$J(v,t,x) := \mathbb{E}^{x,v}\left[m(X_T) + \int_t^T c(X(s))\,ds + \frac{1}{2\beta}\int_t^T v^2(s,X(s))\,ds\right],$$

$$\Phi(t,x) := \inf_v J(v,t,x).$$

- Classical solution: solve Hamilton-Jacobi-Bellman equation

$$\begin{cases} -\frac{\partial \Phi}{\partial t} = \frac{1}{2}\sigma^2(x)\frac{\partial^2 \Phi}{\partial x^2} + \inf_v \left[(b(x)+\sigma(x)v)\frac{\partial \Phi}{\partial x} + c(x) + \frac{1}{2\beta}v^2\right], \\ \Phi(T,x) = m(x). \end{cases}$$

- Transformation $\Phi(t,x) = -\frac{1}{\beta}\ln\psi(t,x)$ gives linear PDE

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- 'Path integral approach' [Kappen, 2005]: approximate ψ by Monte Carlo methods.

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KL weighted optimization of diffusions III - probabilistic approach

- Recall from introduction that $Z^* \propto \exp(-\beta C)$ is optimal density.
- 'Unfortunately' restricted to densities of form

$$Z^v = \exp\left(\int_0^T v(s)\, dW(s) - \frac{1}{2}\int_0^T v(s)^2\, ds\right). \qquad (\bigstar)$$

Theorem

For every density Z there exists a v such that $(\bigstar)$ holds.

'Proof':

Use martingale representation theorem for Z. □

- So may obtain value function by sampling from X, and weighing by $\exp(-\beta C)$, *regardless of dynamics of X and cost structure C*

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Example

- $(X(t))_{0\le t\le T}$ denote a standard Brownian motion.
- $C := \frac{1}{2}(X_T - x^*)^2$,
- $Z^* \propto \exp(-\beta C) = \exp\left(-\frac{1}{2}\beta(X_T - x^*)^2\right)$.
- $$Z^*(t) = \mathbb{E}[Z^*|X(t)] = \frac{1}{K\sqrt{\rho(t)}}\exp\left(-\tfrac{1}{2}\beta(X(t) - x^*)^2/\rho(t)\right)$$
  where $\rho(t) := 1 + \beta(T - t)$
- Using Itô's formula
  $$dZ^*(t) = -\frac{\beta(X(t) - x^*)}{\rho(t)} Z(t)^*\, dX(t).$$
- Apparently, for $u^*(t, X(t)) = -\frac{\beta(X(t)-x^*)}{\rho(t)}$, $0 \le t \le T$, we have $\frac{d\mathbb{P}^{u^*}}{d\mathbb{Q}} = Z^*$.
- The process u^* minimizes
  $$J(u) = \mathbb{E}^u\left[\tfrac{1}{2}(X_T - x^*)^2 + \frac{1}{2\beta}\int_0^T u(s)^2\, ds\right].$$

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Solution using Malliavin calculus

Malliavin derivative

- $(\Omega, \mathbb{P})$ a probability space.
- X a random variable.
- The Malliavin derivative $t \mapsto D_t X$ of X is 'differentiation with respect to ω'.
- If $X = \int_0^T f(s)\, dW(s)$ with f deterministic, $D_t X = f(t)$.

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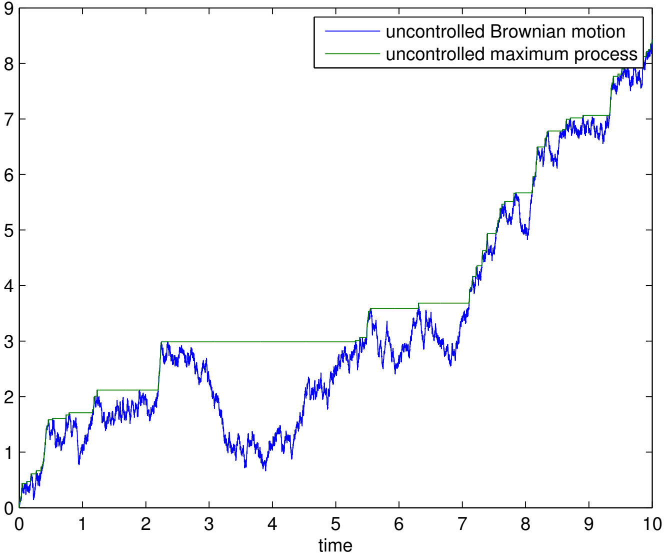
Running maximum $M(t) = \max_{0 \le s \le t} W(s)$



Example: $C = \max_{0 \le t \le T} W(s)$

- Theorem: Optimal control $u^*(t)$ is given by the Malliavin derivative $D_t \ln Z(t)^*$.
- Explicit expressions for $Z^*(t)$ and $D_t Z^*(t)$ are available.
- $u(t)^* = D_t \ln Z^*(t) = \frac{D_t Z^*(t)}{Z^*(t)}$.
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Solves the optimization problem

$$\text{minimize} \quad J(u) := \mathbb{E}^u\left[\max_{0 \le t \le T}\left(W^u(t) + \int_0^t u(s)\, ds\right) + \frac{1}{2\beta}\int_0^T u(s)^2\right],$$

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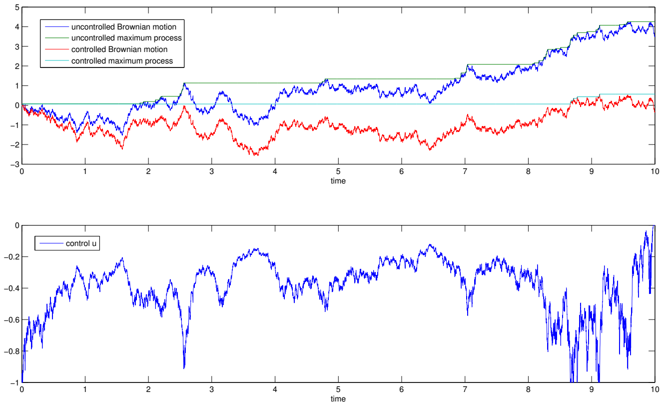

Summary of KL-weighted optimization of diffusion

- General dynamics for X (may be extended to general Markov processes)
- General cost function C
- No HJB PDE involved
- Explicit solution may be obtained by using Malliavin calculus.

References

- H.J. Kappen, *Linear theory for control of nonlinear stochastic systems*, Physical review letters, 2005
- J. Bierkens, H.J. Kappen, *Probabilistic solution of relative entropy weighted control*, arxiv, 2012

Thank you!