The Path Taken for k-Path
The k-Path Problem
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Directed

Weighted
Recursively build paths:

$O(n^k)$

Can there be algorithms with run times on the form

$f(k)n^{O(1)}$?

... and if so, how small can $f(k)$ be?
Intuition for FPT

Consider regular graphs of degree $d$:

Either $d > k$:
   There must be $k$-path.

Or $d \leq k$:
   We can list all potential $k$-paths in $nd^k \leq nk^k$ time.
Monien 1985

\[ F_{k-1}, F_{k-2}, \ldots, F_{k-n} \]  \rightarrow  \[ F_k, F_{k-1}, \ldots, F_{k-n} \]
Monien 1985

- 1-2-3-5
- 1-3-4-5
- 1-4-3-5
- 1-4-6-5
- 2-1-3-5
- 2-1-4-5
- 2-3-4-5
- 3-1-4-5
- 3-4-6-5
- 4-1-3-5

k!
Monien 1985 Path Representatives

- 1-2-3-5
- 1-3-4-5
- 1-4-3-5
- 1-4-6-5
- 2-1-3-5
- 2-1-4-5
- 2-3-4-5
- 3-1-4-5
- 3-4-6-5
- 4-1-3-5
Monien 1985

\[ R_{i-1,k-i+1}, R_{i-1,k-i+1}, \ldots, R_{i-1,k-i-1} \]

\[ R_{i,k-i}, R_{i,k-i}, \ldots, R_{i,k-i} \]
Bodlaender 1989

DFS

1985 1989

\[ 2^k k! \]
Prob[rainbow k-path] >= k!/k^k \sim e^{-k}
Kneis et al. & Chen et al. 2006

Prob[ k-path split] >= 2^{-k}
Kneis et al. & Chen et al. 2006
Combinatorial methods: Tries to construct the object explicitly piece-by-piece.

Algebraic methods: Implicitly sieves for the object by evaluating a sum.

Example: Triangle detection in a graph via fast matrix multiplication.
Prob[ k-path linear independent] >= 1/4
Prob[ k-path linear independent] >=
1 (1-1/2) (1-1/4) (1-1/8) >= 0.28879
• For each vector $s$ in $\{0,1\}^k$, look at the subgraph induced by $V_s$: the vertices whose vectors $v_i$ obey $\langle v_i, s \rangle = 0$.

• Count the $k$-walks in $G[V_s]$, call the result $w_s$.

• Sum over all $w_s$, if sum odd report existence of $k$-path.
Koutis 2008

No False Positives!

k-walk: 1, 2, 3, 1

s =
0 0
0 1
0 1
0 1

2.83^k
Problem: Often False Negatives!

It doesn’t work when all $k$-sized vertex sets induce a graph with an even number of Hamiltonian cycles!
Koutis 2008

Fix: Increase dimension and label edges as well...

$k$-walk: $1,2,3,4$
Schwartz-Zippel Lemma: Let \( P(x_1, x_2, \ldots, x_n) \) be a multivariate non-zero polynomial of total degree \( d \) over a finite field \( F \). For uniformly and independently sampled values \( r_1, r_2, \ldots, r_n \) in \( F \):

\[
\Pr[P(r_1, r_2, \ldots, r_n) = 0] \leq \frac{d}{|F|}
\]
Use larger field of characteristic two!
Counting k-Paths

- [Flum and Grohe 2002] Counting the number of k-paths is \#W[1]-Hard.
- Approximating the number of k-paths within an arbitrarily small constant using the color-coding technique requires at least

\[ \Omega(n^{k/2}) \]

time.
Vassilevska-Williams 2009

(2n)^{k/2}
T(S)=Number of \( \frac{k}{2} \)-paths containing the vertices \( S \).

\[ \sum_{S \in 2^{[n]}} (-1)^{|S|} T(S)^2 \]

\( |S| = \frac{k}{2} \)
Thank you for listening!