# Quantum information and graphs

*Case studies: channels, transport, representations*

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# Plan

- **Channels**: graphs as *abstract objects* corresponding to communication channels
  - zero-error information theory
- **Transport**: graphs as *concrete networks* whose nodes correspond to particles
  - transfer
  - control
- **Representations**: graphs as *abstract object* encoding a physical state
  - graph laplacians

# Channels

# One-shot zero-error capacity

- A **discrete memoriless stationary channel** (DMSC) with alphabet $\{1, ..., n\}$ is represented by a stochastic matrix $M$ such that

$$[M]_{i,j} = \Pr[i \text{ received when } j \text{ transmitted}].$$

- Two symbols $j$ and $k$ are **confusable** if there is $i$ such that $[M]_{i,j}[M]_{i,k} > 0$.
- The **confusability graph** $G = (V, E)$ of the DMSC has $V = \{1, ..., n\}$ and $\{j, k\} \in E$ iff $j$ and $k$ are confusable.
- A set $S \subseteq V$ is an **independent set** of $G$ if $\{i, j\} \notin E$ for every $i, j \in S$. The **independence number** $\alpha(G)$ of $G$ is the cardinality of the largest independence set in $G$.
- The **one-shot zero-error capacity** $C_{s,0}(\mathcal{K})$ of a DMSC $\mathcal{K}$ is the maximum number of symbols that can be transmitted without confusion in one use of the channel.(We can just write $C_{s,0}(G)$.)

## Fact

$C_{s,0}(G) = \log \alpha(G)$ *and this is NP-hard to compute.*

# Zero-error capacity

- The **strong product** of $G$ and $H$ is the graph $K = G \boxtimes H$ with $V(G \times H) = V(G) \times V(H)$ and $\{(i,j),(k,l)\} \in E(K)$ *iff* $\{i,k\} \in E(G)$ AND $\{j,l\} \in E(H)$ or $\{i,k\} \in E(G)$ AND $j = l$ or $\{j,l\} \in E(H)$ AND $i = k$. Let $G^{\boxtimes k} := \underbrace{(G \boxtimes \cdots \boxtimes G)}_{k \text{ times}}$.
- The **zero-error capacity**[2] of $G$ is defined as $C_0(G) = \lim_{k\to\infty} \frac{1}{k} \log \alpha\left(G^{\boxtimes k}\right) = \sup_k \frac{1}{k} \log\left(G^{\boxtimes k}\right)$.

## Fact

*(1)* $C_{s,0}(G) \leq C_0(G)$*; (2)* $C_0(G)$ *not known to be computable.*

- The **chromatic number** $\chi(G)$ of $G$ is the is the minimum number of colours for the vertices such that every two adjacent vertices have a different colour. A graph $G$ is **perfect**[3] if $\chi\left(\overline{H}\right) = \alpha(H)$, for every induced subgraph $H$.
- If $G$ is perfect then $C_{s,0}(G) = C_0(G) = \log \alpha(G) = \log \chi\left(\overline{G}\right)$.

[2] Shannon (1956)

[3] Berge (1973); Chudnovsky, *et al.* (2002)

# Lovász function

- An **orthogonal representation** of $G = (V, E)$ is an assignment of vectors $|1\rangle, ..., |n\rangle$ to $V = \{1, ..., n\}$ such that $\langle i|j\rangle = 0$ iff $\{i, j\} \in E$. Let $\{|v_1\rangle, ..., |v_n\rangle\}$ range over all orth. repr. of $\overline{G}$ and $|\psi\rangle$ over $\mathbb{R}^d$. The **Lovász $\vartheta$-function**[4] is

$$\vartheta(G) = \max \sum_{i=1}^{n} |\langle\psi|v_i\rangle|^2.$$

- (1) $\vartheta(G \boxtimes H) = \vartheta(G)\,\vartheta(H)$; (2) $\vartheta(G)$ is an SDP relaxation; (3) **there are graphs such that** $C_0(G) < \vartheta(G)$.

## Theorem (Sandwich Theorem)

$\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G})$.[5]

## Example

$C_0(C_5) = \frac{1}{2}\log 5$. Take $\{|v_1\rangle, ..., |v_5\rangle \in \mathbb{R}^3\}$ *s.t.*
$v_i = \left(\cos\phi\cos\frac{2\pi j}{5}, \cos\phi\sin\frac{2\pi j}{5}, \sin\phi\right)^T$, where
$\phi = \tan^{-1}(-\cos 4\pi/5)^{1/2}$.

[4] Lovász (1979)
[5] See, Knuth (1994)

# Non-commutative graphs (1/2)

- The parties can **share** a quantum state. Systems with **two parties** have a space $\mathcal{H}_{AB} \cong \mathbb{C}^n_A \otimes \mathbb{C}^m_B$.
- A Hermitian $\rho_{AB}$ on $\mathcal{H}_{AB}$ such that $\mathrm{Tr}(\rho_{AB}) = 1$ and $\rho_{AB} \succeq 0$ is a state. If we cannot write $\rho_{AB} = \sum \omega_i \rho_A^{(i)} \otimes \rho_B^{(i)}$, $\sum \omega_i = 1$, then $\rho_{AB}$ is said to be **entangled**.
- A **quantum channel** is a map $\mathcal{K}(\rho) = \sum_j E_j \rho E_j^\dagger$, where the $E_j$'s are (Kraus) Hermitian operator such that $\sum_j E_j E_j^\dagger = I$.
- Every classical channel is also quantum.
- The **non-commutative (confusability) graph** associated to a quantum channel $\mathcal{K}$ is the operator subspace $S = \mathrm{span}\{E_j^\dagger E_k : i, j\}$, where $E_j$ are the Kraus operators of $\mathcal{K}$.[6]

[6] Duan-Paulsen-Severini-Todorov-Winter (in preparation); another approach Kuperberg-Weaver (2011)

# Non-commutative graphs (2/2)

- An operator space $S$ is associate to a channel iff $I \in S$ and $S = S^\dagger$.
- The max number of one-shot zero-error distinguishable messages for $\mathcal{K}$ (the **independence number** $\alpha(S)$ of $S$) is the max size of a set of orthogonal vectors $\{|\phi_m\rangle : m = 1, ..., n\}$ such that $\forall m \neq m'$, $|\phi_m\rangle\langle\phi_{m'}| \in S^\perp$: input states $|\phi_m\rangle$ and $|\phi_{m'}\rangle$ lead to orthogonal output states iff $0 = \mathrm{Tr}\mathcal{K}(\phi_m)\mathcal{K}(\phi_{m'}) = \sum_{j,k} |\langle\phi_{m'}|E_j^\dagger E_k|\phi_m\rangle|^2$. Computing $\alpha(S)$ is QMA-complete and NP $\subseteq$ QMA.
- For a **(classical) confusability graph** $G$, $S = \{T : \forall\{i,j\} \notin E(G), \langle i|T|j\rangle = 0\}$.[7]
- When the parties share an entangled state, the **entanglement-assisted zero-error capacity** of $G$ is $C_{s,e,0}(G) = \widetilde{\alpha}(G)$; asymptotically, $C_{e,0}(G) = \sup_k \frac{1}{k} \log\left(G^{\boxtimes k}\right)$.

[7] Rank problems; Colin de Verdière graph invariant, *etc.*

# Results

- Let $G$ be a graph with $V$ partitioned into $k$ $d$-cliques $B_1, ..., B_k$. We label each vertex of $G$ by $(q, i)$, where $q \in [k]$ and $i \in [d]$. The graph $G$ realizes a (weak) **Kochen-Specker set**[8] if it has an orthogonal representation $\{|q, i\rangle : (q, i) \in V\}$ of dimension $d$ and $G$ **does not have** a $k$-clique $\{v_1, ..., v_k\}$, where $v_i \in B_i$.

## Theorem

*If $G$ realizes a Kochen-Specker set then $C_{s,0}(G) < \log k = C_{s,e,0}(G)$.*[9]
*[Proof: explicit protocol]*

## Theorem

*There are graphs such that $C_0(G) < C_{e,0}(G)$.*[10] *[Proof: some finite field graphs]*

## Theorem

$C_{e,0}(G) \leq \vartheta(G)$.[11] *[We conjecture equality]*

[8] Kochen-Specker (1967)
[9] Cubitt-Leung-Matthews-Winter protocol (2010)
[10] Leung-Mancinska-Matthews-Ozols-Roy (2010)
[11] Duan-Severini-Winter (2010)

# Transport

# The framework

- The **Hamiltonian** for a system of 2-dim particles on a graph $G$ has Hilbert space $\mathcal{H} \cong \mathbb{C}_1^2 \otimes \cdots \otimes \mathbb{C}_n^2$. (under mild restrictions) This is

$$\widehat{H} = \frac{1}{2} \sum_{\{i,j\} \in E} J_{ij} \left( \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y \right),$$

  where $\sigma_u^w$ ($w \in \{x, y, z\}$) is a Pauli matrix on $\mathbb{C}_u^2$; $J_{ij}$ the coupling strength.
- **Continuous-time quantum walks:** by the Jordan-Wigner transform, the free evolution for a time $t$ of a single **excitation** from site $|u\rangle \in \{|1\rangle, \ldots, |n\rangle\}$ is given by $e^{iAt}|u\rangle = U(t)|u\rangle$, where $A$ is the adjacency matrix of $G$.
- This framework models **state transfer** in spin systems, **information transfer** in optical guides, **energy transport** in molecular complexes (without added noise).

# Perfect state transfer and periodicity

- There is **perfect state transfer (PST)** in $G$ between $i$ and $j$ if there is $t$ *s.t.* $|\langle j|U(t)|i\rangle| = 1$.[12]

## Theorem

*(1) Hypecube have PST between antipodal vertices for* $t = \pi/2$*;*
*(2) n-paths have PST only if* $n = 2, 3$*; (3) Cayley graphs of* $\mathbb{Z}_2^k$ *have PST iff the sum of the Cayely elements is not the id.*

- A graph $G$ is **periodic** if there is $t$ *s.t.* $|\langle i|U(t)|i\rangle| = 1$ for every $i$.

## Theorem

*If G is a connected regular graph, then it is periodic iff its eigenvalues are integers.*

- There is **pretty good state transfer (PGST)** in $G$ between $i$ and $j$ if for every $\epsilon > 0$ there is $t$ *s.t.* $|\langle n|U(t)|1\rangle| > 1 - \epsilon$.

## Theorem

*There is PGST in the n-path iff* $n = p - 1$ *or* $2p - 1$*, where p is a prime, or if* $n = 2^m - 1$*.*

[12] Bose (2003), Christandl-Datta-Ekert-Landahl (2003), Saxena-Severini-Shparlinski (2007), Godsil (2009-2012), *et al.*

# Mixing

- Define a **uni-stochastic** matrix $M(t) := U(t) \circ U(-t)$.
- The **average mixing matrix** is $\widehat{M}(G) := \lim_{T\to\infty} \frac{1}{T}\int_0^T M(t)dt = \sum_r E_r^{\circ 2}$, where each $E_r$ is an idempotent in the spectral decomposition of the adjacency matrix. *E.g.*, for the $n$-cycle with $n$ odd $\widehat{M}(C_n) = \frac{n-1}{n^2}J + \frac{1}{n}I$, where $J$ is the all-ones matrix.

## Theorem

*The average mixing matrix of a graph is rational.*[13]

- We have **uniform mixing** at time $t$ if $U(t)$ is **flat**, *i.e.*, all entries of $U(t)$ have the same absolute value.

## Theorem

*If uniform mixing occurs on $C_n$ then $4|n$ and $n$ is the sum of two squares.*

## Problem

*Determine which graphs have uniform mixing.*

[13] Godsil & collaborators

# Control

- Let $G$ be a graph on $n$ vertices and let $z \in \mathbb{R}^n$. Let $W_z := \begin{pmatrix} z & Az & \dots & A^{n-1}z \end{pmatrix}$ an $n \times n$ matrix with entries in $\mathbb{Z}^{\geq 0}$ associated to $G$. When $z$ is the characteristic vector of some set $S \subseteq V$, the matrix $W_S$ is called a **walk matrix** of $G$ with respect to $S$.
- The pair $(G, S)$ is said to be **controllable** if the matrix $W_S$ is invertible (*i.e.*, $\det(W_S) \neq 0$). A graph $G$ is said be **controllable** if $(G, \mathbf{1})$ is controllable.
- A controllable graph can not be regular.

## Theorem

*If $(G, S)$ is a controllable pair then the unitary matrices $U_A(s) = e^{-iAt}$ and $U_S(t) = e^{-iss^T t'}$, $t, t' \in \mathbb{R}^{\geq 0}$, generate a dense subgroup of the unitary group.*

## Problem

*A.a.s. a graph is controllable.*

# Representations

# Laplacians as quantum states

- A **quantum state** is a matrix $\rho$ such that (1) $\rho = \rho^{\dagger}$, (2) $\rho$ is positive semidefinite, (3) $\rho$ has trace 1.
- Let $G$ be on $n$ vertices and $m$ edges. Two configuration (Hilbert) spaces: $\mathcal{H}_V \cong \mathbb{C}^V$ with orthonormal basis $\widehat{\mathbf{a}}_v$, $v \in V$; $\mathcal{H}_E \cong \mathbb{C}^E$ with orthonormal basis $\widehat{\mathbf{b}}_e$, $e \in E$.
- The **Laplacian** of $G$ is $L = D - A$ acting on $\mathcal{H}_V$, where $D_{i,j} = d(i) - \delta_{ij}$; $L = M_{\mathbf{f}} M_{\mathbf{f}}^T$, where $M_{\mathbf{f}}$ is the incidence matrix of any orientation of $G$. Then $\rho := \frac{1}{TrL} L$ is a quantum state and we can treat it as such!
- We call $\Psi_{\mathbf{f}} = \sum_{uv \in E} f_{uv}(u) \left(\widehat{\mathbf{a}}_u - \widehat{\mathbf{a}}_v\right) \otimes \widehat{\mathbf{b}}_{uv} \in \mathcal{H}_V \otimes \mathcal{H}_E$, **incidence vector**.

## Fact

*Let $\Psi_{\mathbf{f}}$ be any incidence vector of a graph $G$. Then $L(G) = tr_E\left(\Psi_{\mathbf{f}} \Psi_{\mathbf{f}}^{\dagger}\right)$ and $L_E(G) = tr_V\left(\Psi_{\mathbf{f}} \Psi_{\mathbf{f}}^{\dagger}\right) = M_{\mathbf{f}}^T M_{\mathbf{f}}$.*

## Fact

*The* **Schmidt rank** *of $\Psi_{\mathbf{f}}$ is $n - w(g)$, the number of connected components of $G$.*

# Entropies

- The **von Neumann entropy** is defined as $S(\rho) = -\sum \lambda_i \log_2 \lambda_i$, where $\lambda_i$ is the $i$-th eigenvalue of $\rho$. Then $S(\rho)$ is the amount of **entanglement** between $\mathcal{H}_V$ and $\mathcal{H}_E$!
- Each edge can be interpreted as a pure state and the laplacian is a **statistical mixture**...generalizing a probability distribution.
- Given density matrices $\sigma_1$ and $\sigma_2$, the **quantum relative entropy** of $\sigma_1$ with respect to $\sigma_2$ measures the difficulty of distinguishing between these states and it is defined by $S(\sigma_1 \| \sigma_2) = \mathrm{Tr}(\sigma_1 \ln \sigma_1) - \mathrm{Tr}(\sigma_1 \ln \sigma_2)$. The number of spanning trees of a graph can be computed by making you of this quantity.

## Problem

*What is the interpretation of other graph-theoretic quantities? We can transform a graph into cospectral ones by evolving unitarily its state. Can we characterize the operations that preserve entanglement?* **What about mixtures of graphs, Holevo information, etc. ?**

# Conclusions

- **Channels**: We have a (physical?) interpretation for the Lovász $\vartheta$-function. We have a generalization of combinatorial notions to the operator algrebraic framework, by looking at graphs as operator spaces.
- **Transport**: Lots of ways to look at graphs/networks as physical systems. New graph-theoretic parameters. New mathematical challenges (*e.g.*, modeling the plants power grids).
- **Representations**: A framework to re-think laplacians and to employ a well-established mathematical machinery in network analysis.
- **Extra:** Non-contextuality, nonlocal games, Hamiltonian for the evolution of time-dependent graphs, Lieb-Robinson-type bounds, graph states, dynamics of entanglement in networks, quantum "random graphs" models, *etc.*

# References


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