Polymatroids and Submodularity

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- Linear Programming Philosophy
- The Polymatroid Theorem (1)
- The Greedy Alg Theorem (2)
- The Intersection Theorem (3)
- The Partition Theorem (4)
- Opt Branching Systems Application

A predicate p(x),

i.e., a statement with a variable input subject, x,

is called NP if whenever it is true

there is an easy proof (certification) of it

(polynomial time relative to the size of x).

p(x) is called coNP when ‘not p(x)’ is NP.

A theory is called a ‘good characterization’ of p(x)

when it states how p(x) is in NP∩coNP.

It is a practical conjecture that if p(x) is in NP∩coNP

then there is a polytime algorithm for deciding it.

Linear program: Maximize cx where $x \geq 0$ and $Ax \leq b$.

Dual linear program: Minimize yb where $y \geq 0$ and $yA \geq c$.

Duality Theorem: max = min.

Polytope: a bounded convex polyhedron, $P = \{x : x \geq 0 \text{ and } Ax \leq b\}$.

Or: all convex combinations of a finite set V.

Caratheodory Theorem. x in P iff

convex combination of a small number from V.

An easy description of system $L = \{ x \geq 0 \text{ and } Ax \leq b\}$

and an easy description of V,

provides NP∩coNP,

and probably polytime algorithms.

There are several known classes of "nice systems", L = { $x \geq 0$ and $Ax \leq b$}, where the matrix A is all 0s & 1s. Let P_L = {x satisfying L}.

Then where $S \subseteq E$ (≡ the column index set),

$f(S) = \max [x(S)] \equiv \max [\sum x_j : j \text{ in } S]$, $x \in P_L$, is called the set extension.

In fact for $w \in R^E_+$, $f(w) = \max \{wx: x \in P_L\}$, is called the real extension.

Obvious: $P_L = P(E, f) \equiv \{x : x \geq 0; x(S) \leq f(S) \text{ for every } S \subseteq E\}$.

Polymatroids turn out to be P(E, f) where the set function, f, is submodular, non-decreasing, and $f(\phi) = 0$.

For most known classes of nice systems, the f has not been studied.

Example Theorem: For polymatroid functions f_1 & f_2 :

Minkowski sum, $P(E,f_1) + P(E,f_2) = P(E, f_1 + f_2)$. **Other systems?**

You know: For any finite set E of vectors over any field, every maximal independent subset (basis of E) is a largest independent subset.

MATROID, M = (E, F):

Abstract set E with “independent” subsets

so that every subset of an ind set is ind; and

For every subset S of E, every maximal independent subset of S (basis of S) is largest,

called the rank “r(S)”.

Let F denote the family of independent sets.

An **integer** **polymatroid** F with groundset E
is a bounded set F of non-negative integer-valued vectors
coordinatized by E

such that F includes the origin; and
any non-neg integer vector is a member of F
if it is ≤ some member of F; **and**
for any non-neg integer vector y,
every maximal integer vector x in F which is ≤ y
has the same sum $\mathbf{x(E) \equiv \Sigma(x_j : j \in E)}$**,**
called the rank **r(y)** of vector y in F.

The 0–1 vectors in an integral polymatroid are the incidence vectors
of the independent sets J (i.e., J ∈ F) of a matroid M = (E,F).

Polymatroid: leave out "integer".

For any vectors x and y in Z_+,
$x \cap y$ denotes the maximal integer vector which is ≤ x and ≤ y;
$x \cup y$ denotes the minimal integer vector which is ≥ x and ≥ y.
Clearly $0 \le r(0) \le r(x) \le r(y)$ for $x \le y$ *(non-neg & non-decreasing); AND*

Proposition. For any x and y in Z_+, **$r(x \cup y) + r(x \cap y) \le r(x) + r(y)$.** ***(submodularity)***

Proof: Let v be an P-basis of $x \cap y$. Extend it to P-basis w of $x \cup y$. $r(x \cap y) = |v|$ and $r(x \cup y) = |w|$.
$r(x) + r(y) \ge |w \cap x| + |w \cap y| = |w \cap (x \cap y)| + |w \cap (x \cup y)| = |v| + |w| = r(x \cup y) + r(x \cap y)$. □

For subsets S of E, define $f_F(S)$ to be r(x) where $x \in Z_+$ is very large for coordinates in S and = 0 for coordinates not in S.

Corollary. For sets $S \subseteq E$, $f_F(S)$ is a non-neg, non-decreasing, and submodular set function.

Such an $f_F(S)$ for subsets $S \subseteq E$, is called a ***polymatroid function***.
(We will see how, conversely,
any polymatroid is given by a polymatroid function.

Discussion of ***polymatroid*** is the same by omitting everywhere the word 'integer'

(but including only the condition that it be closed and bounded**)**.

It will be theorem that any polymatroid F is in fact a polytope.

However, when it is an integral polymatroid,
$f_{F(S)}$ is integer valued
and $F = \{x\colon x \geq 0;\ x(S) \leq f_{F(S)}\}$ has *total dual integrality* (TDI),

meaning F has integer-valued vertices (primal integrality),

and for any integer-valued c,
the primal l.p., $\max\{cx\colon x \geq 0;\ x(S) \leq f_{F(S)}\}$, and
the dual l.p., $\min\{\sum y_S f_{F(S)}\colon y_S \geq 0;\ \sum(y_S : S \text{ containing } j) \geq c_j \}$,
have integer-valued optima $x = (x_j : j \text{ in } E)$ and $y = (y_S : S \subseteq E)$.
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polymatroid function: A real-valued function **$f(S)$, $S \subseteq E$**, where
(a) $f(\emptyset) = 0$ and $f(A) \geq 0$ for every $A \subseteq E$;
(b) is non-decreasing: $A \leq B \Rightarrow f(A) \leq f(B)$; and
(c) submodular: $\mathbf{f(A \cup B) + f(A \cap B) \leq f(A) + f(B)}$ for $\mathbf{A \subseteq E}$ and $\mathbf{B \subseteq E}$.

1) The Polymatroid Theorem. For a polymatroid function, $f(S)$, $S \subseteq E$, the following polyhedron is a polymatroid:
$P(E, f) \equiv \{ x \geq 0 : x(S) \leq f(S)$ for every $S \subseteq E\}$.
The polymatroidal rank function r of $P(E, f)$, for any $a = [a_j : j \in E] \geq 0$, is $r(a) = \min [f(S) + a(E - S) : S \subseteq E]$.

Proof. Clearly $r(a) \leq \min [f(S) + a(E - S) : S \subseteq E]$ (weak l.p. duality).
Let $x_0 \geq 0$ be any maximal $x \geq 0$ such that $x \leq a$ and $x(S) \leq f(S)$ for every $S \subseteq E$.
For any sets A and B such that $x_0(A) = f(A)$ and $x_0(B) = f(B)$, we have
$x_0 (A \cup B) + x_0 (A \cap B) \leq f(A \cup B) + f(A \cap B)$
$\leq f(A) + f(B) = x_0(A) + x_0(B) = x_0 (A \cup B) + x_0 (A \cap B)$,
and so $x_0 (A \cup B) = f(A \cup B)$ and $x_0 (A \cap B) = f(A \cap B)$,
and so where S is the union of all sets where equality holds,
we have $x_0(S) = f(S)$ and $x_0(E - S) = a(E – S)$. Hence $r(a) = f(S) + a(E – S)$.
By the same argument, where f is integer–valued we get an integer polymatroid.

Theorem. For any polymatroid P: we have P = P(E, f_P).
Thus, all polymatroids are polytopes,
and they correspond precisely to polymatroid set functions.

Submodular set function minimization:

Any submodular set function f'(S), S ⊆ E,
not necessarily non-neg and non-decreasing,
can, for some easily specied vector a ≥ 0,
be expressed up to an additive constant as
f' = f(S) + a(E - S) where f(S) is a polymatroid function.

And so Thm 1 again: r(a) = min [f'(S) : S ⊆ E].

That is, if we have an oracle which tells the value of f'(S) for any S ⊆ E,
we can **find the minimum of f'(S)**
by getting a maximal vector x
such that x ϵ P(E, f) ≡ { x ≥ 0 : x(S) ≤ f(S) for every S ⊆ E}
and x ≤ a.

The only difficulty in simply pushing x up
arbitrarily in the various coordinates, starting with x = 0,
until any further push
will violate x ≤ a in some coordinate or x(S) ≤ f(S) for some S ⊆ E,
is that **there are too many inequalities, x(S) ≤ f(S), to check.**

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When f is the rank function of a matroid, x(J) = f(j) iff J is independent.

More generally, it is easy to verify that an x **is not in P(E, f)**
i.e., not independent,
by showing x(S) > f(S) for one set S ⊆ E.
But how can we verify that x satisfies x(S) ≤ f(S) for all S ⊆ E?

We get [min f'(S)] into NP∩coNP relative to the oracle for f'(S)
by getting a good description of when x **is in P(E, f).**
We do that by getting a good description of the vertices of **P(E, f).**

Then we can certify that x ϵ P(E, f) by using Cartheodory's theorem to show x
as a convex combination of a small number of vertices of **P(E, f).**

Let j(1), j(2), …, j(k), be an arbitrary ordering of a subset of E.
For each integer i, $1 \le i \le k$, let $A_i = \{j(1), j(2), . . . , j(i)\}$.

2) Greedy Vertex Theorem.

The vertices of P(E, f) are the vectors, x, where

$x_{j(1)} = f(A_1)$;

$x_{j(i)} = f(A_i) - f(A_{i-1})$ for $2 \le i \le k$; and $x_{j(i)} = 0$ for $k < i \le |E|$.

To solve the linear program, maximize $cx = (\Sigma c_j x_j : j \in E)$ over $x \in P(E, f) \equiv \{ x \geq 0 : x(S) \leq f(S)$ for every $S \subseteq E\}$, let $j(1), j(2), \ldots$ be an ordering of E such that $c_{j(1)} \geq c_{j(2)} \geq \cdots c_{j(k)} > 0 \geq c_{j(k+1)} \geq \cdots$. For each integer i, $1 \leq i \leq k$, let $A_i = \{j(1), j(2), \ldots, j(i)\}$.

The dual l.p. is to Minimize $f\,y = \Sigma[\, f(S)y(S): S \subseteq E]$ where $y(S) \geq 0$; and for every $j \in E$, $\Sigma\,[y(S): j \in S] \geq c_j$.

2') The Greedy Algorithm:

cx is maximized over $x \in P(E, f)$ by the vector x of the form in Theorem 2:
$x_{j(1)} = f(A_1)$; $x_{j(i)} = f(A_i) - f(A_{i-1})$ for $2 \leq i \leq k$; and $x_{j(i)} = 0$ for $k < i \leq |E|$.

The Dual Greedy Algorithm:

An optimum solution, $y = [y(A)]$, $A \subseteq E$, to the dual l.p.is
$y(A_i) = c_{j(i)} - c_{j(i+1)}$ for $1 \leq i \leq k - 1$;
$y(A_k) = c_{j(k)}$; and $y(A) = 0$ for all other $A \subseteq E$.

2'') Where f is the rank function of a matroid M = (E,F), Theorem 2 implies that
the vertices of P(E, f) are precisely the 0,1 incidence vectors of the independent sets J ϵ F of matroid M.
Such a P(E, f) is called a matroid polytope. For matroids, the greedy alg takes a more familiar looking form. (Boruvka 1930s)

2''') Another form of the Greedy Alg Thm.

For any submod, f, with f(∅) = 0, (not non-decreasing), and any $c \in R^E$,

max[cx ≡ ($\Sigma c_j x_j$: j ϵ E) over x ϵ P'(E, f)],

where **P'(E, f) ≡ { x: x(S) ≤ f(S) for every S ⊆ E, x(E) = f(E) },**

is achieved by

$x_{j(k)}$ = f[j(1),…,j(k)] – f[j(1),…,j(k-1)] where $c_{j(1)} \geq c_{j(2)} \geq \ldots c_{j(n)}$.

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f(w) = max[wx over x ϵ P'(E, f)], (= dual l.p. min), $w \in R^E$,
is a real extension of f(S), S ⊆ E, sometimes called the Lovasz extension of f(S).

Optional. The facets of a polymatroid.

Let f be a polymatroidal function on S ⊆ E.
A set A ⊆ E is called f-closed or an f-flat,
when f(A) < f(C) for any C ⊆ E which properly contains A .

Theorem. If A and B are f-closed then A ∩ B is f-closed.

A set A ⊆ E is called *f-separable* when $f(A) = f(A_1) + f(A_2)$
for some partition of *A* into non-empty subsets A_1 and A_2.
Otherwise *A* is called *f*-inseparable.

Theorem. Any A ⊆ E partitions in only one way
into a family of f-inseparable sets A_i such that $f(A) = \Sigma f(A_i)$.
The Ai's are called the f-blocks of A.

Theorem. Where f is a polymatroidal function
such that the empty set is f-closed,
the facets of polymatroid P(E, f) are:
$x_j \geq 0$ for every j ∈ E;
and $x(A) \leq f(A)$ for every A ⊆ E which is f-closed and f-inseparable.

“The Hitchcock transportation problem”, longtime well-known in O.R.:
Let each Vp, p = 1 and 2,
be a family of mutually disjoint subsets of H.

Where [aij], i ∈ E, j ∈ E, is the 0–1 incidence matrix of $V_1 \cup V_2$,
the following l.p. is known in O.R.
as the

Maximize cx = Σ]$c_j x_j$: j∈E] , where x satisfies
L = [$x_j \geq 0$ for every j ∈ E, and Σ[$a_{ij} x_j$: j∈E] ≤ b_i for every i ∈ H].
The dual l.p. is Minimize by = Σ$b_i y_i$, where
$y_i \geq 0$ for every i ∈ H, and Σ[$a_{ij} y_i$: i∈H] ≥ c_j for every j ∈ E.

The following properties of the Hitchcock problem are important in its combinatorial use. This is essentially Egervary‘s Theorem, 1931.
Theorem. Where the b_i‘s are integers,
the primal l.p. has integer valued opt.
Where the cj ‘s are integers,
the dual l.p. has integer value opt.
In other words, the primal system L is TDI.

3) The Integer Polymatroid Intersection Theorem:
For any two integral polymatroids $P_1 = P(E, f_1)$ and $P_2 = P(E, f_2)$, $P_1 \cap P_2$ is TDI. In particular, the vertices of $P_1 \cap P_2$ are integer-valued.

L.P. Dual of Polymatroid Intersection.
$P(E, f_1)$ and $P(E, f_2)$ are polymatroids (not necessarily integral),
the dual of the l.p.:
Maximize $cx = \Sigma\ [c_j x_j\colon j \in E]$, where $x \in P(E, f_1) \cap P(E, f_2)$
is the l.p.:
Minimize $fy = \Sigma\ [f_1(S)y_1(S) + f_2(S)y_2(S):]$
where for every $S \subseteq E$, $y_1(S) \geq 0$ and $y_2(S) \geq 0$;
and for every $j \in E$, $\Sigma\ [y_1(S) + y_2(S)\colon j \in S \subseteq E] \geq cj$.

TDI: *If the c_j's are all integers, then, regardless of whether f_1 and f_2 are integral, there is an integer-valued solution y which minimizes fy.*

max cx = min fy where $x \in P(E, f_1) \cap P(E, f_2)$
and where y is dual feasible.
If f is integral, x can be integral. If c is integral, y can be integral.

3’) The Matroid Polytope Intersection Theorem.

Where P_1 and P_2 are the polytopes
of any two matroids M_1 and M_2 on E,
the vertices of $P_1 \cap P_2$ are precisely the vectors
which are vertices of both P_1 and P_2
namely, the incidence vectors of sets
which are independent in both M_1 and M_2.
In fact, the inequality system for $P_1 \cap P_2$ is TDI.

Where P_1, P_2, and P_3 are the polytopes of three matroids on E,
polytope $P_1 \cap P_2 \cap P_3$ generally has many fractional vertices
besides those which are vertices of P_1, P_2, and P_3.

Where $c = [c_j]$, $j \in E$,
is any numerical weighting of the elements of E,
Finding a set J, independent in both M_1 and M_2,
that has maximum weight sum, $\Sigma(c_j: j \in J)$,
is equivalent to the *l.p.* problem:
Find a vertex x of $P1 \cap P2$ that maximizes cx.

Assuming there is a good algorithm for recognizing
whether or not a set $J \subseteq E$ is independent in M_1 or in M_2,
there is a good algorithm for this problem.

In particular, where f_1 and f_2 are the rank functions, r_1 and r_2,
of any two matroids, $M_1 = (E,F_1)$ and $M_2 = (E,F_2)$,
and where every $c_j = 1$, we have

3'') The Matroid Cardinality Intersection Theorem:
For any two matroids $M_1 = (E,F_1)$ and $M_2 = (E,F_2)$,
$\max |J : J \in F_1 \cap F_2| = \min [r_1(S) + r_2(E - S): S \subseteq E]$.

Example Application. A branching S rooted at node v_0 in directed graph G(V, E), having node-set V and edge-set E, means a set $S \subseteq E$ such that
1) S is the edge-set of a spanning tree of G, and
2) For each node $v \neq v_0$, exactly one edge of S enters v.

Condition (1) means S is a basis (maximal independent set) in matroid M_1.
Condition (2) means S is a basis in matroid M_2.

So for any weighting $c = (c_j : j \in E)$ finding an optimum branching in G(V, E) means finding an optimum weight common basis of matroids M_1 and M_2 , i.e., an optimum vertex of polytope $P(E, r_1) \cap P(E, r_2)$.

Algorithm sketch:
Choose best edge to each $v \neq v_0$ until a directed cycle is created.
Shrink the cycle to a pseudo-node, and continue. When chosen set not shrunk is a branching in the 'pseudo-node graph', expand the pseudo-nodes in the reverse order they were creating and use chosen edges in the unique way to get a branching in G(E, V).

Using also a 3rd matroid similar to (2) we get the "traveling salesman problem". We don't know a good way to optimize over common bases of 3 matroids.

Matroid Partitioning, another application of Matroid Intersection

Matroid Sums:

Let $\{M_i: i \in y I\}$, be a family of matroids, $M_i = (E,F_i)$, having rank functions r_i.
Let $J \subseteq E$ be a member of F iff $|S| \leq \Sigma(r_i(S): i \in I)$ for every $S \subseteq J$.
Since $f(S) = \Sigma[r_i(S): i \in I]$ is an integer polymatroid function on $S \subseteq E$,
$M = (E,F)$ is a matroid, called the sum of the matroids M_i.

4) The Matroid Partitioning Theorem (also called The Matroid Union Thm):

$J \in F$ iff J can be partitioned into sets J_i such that $J_i \in F_i$.

Can be proved by a polytime algorithm which either finds a partitioning of J into sets $J_i \in F_i$ or else finds one set $S \subseteq E$ such that $|S| > \Sigma(r_i(S): i \in I)$.

Since $M = (E,F)$ is a matroid we can use the greedy algorithm to find an optimum weight set J which is partitionable into sets independent respectively into given matroids M_i.

In particular given graphs $G_i(V_i , E)$ we can easily find an optimum subset of E which is partitionable into forests, respectively, of the graphs $G_i(V_i , E)$.

Proof of (4) from The Matroid Intersection Theorem:

Obtain for a given family of matroids $M_i = (E, F_i)$, $i \in I$, an "optimum" family of mutually disjoint sets $J_i \in F_i$, by using the matroid intersection algorithm (3'') on the following two matroids $M_1 = (E_I, F_1)$ and $M_2 = (E_I, F_2)$.

Let E' consist of all pairs (j, i), $j \in E$ and $i \in I$.

Let $M_1 = (E', F_1)$ be the matroid such that $J \subseteq E'$ is a member of $F1$ iff the corresponding sets J_i are mutually disjoint — that is, if and only if the j's of the members of J are distinct.

Let $M_2 = (E', F_2)$ be the matroid such that $J \subseteq E'$ is a member of F_2 iff the corresponding sets J_i are such that $J_i \in F_i$.

The Minkowski Sum Theorem for polymatroids:

$P(E,f_1) + P(E,f_2) = P(E, f_1 + f_2)$

can be obtained from the intersection of two polymatroids in the same way.

Optimum System of k edge-disjoint branchings

in digraph G(V,E), rooted at node v_0 :

an optimum subset $J \subseteq E$, where J is a basis of matroids

$M_1 = (E,F_1)$ and $M_2 = (E,F_2)$,

i.e., $J \in F_1 \cap F_2$ and $|J| = r(E)$,

where for the two matroids $M_1 = (E,F_1)$ and $M_2 = (E,F_2)$,

J is a basis of matroid M_1

when J can be partitioned into k spanning trees of G(V,E);

and J is a basis of matroid M_2

when exactly k edges of J enter each node $\neq v_0$.

A version of the 'Disjoint Branchings Theorem':

J can then be partitioned into k edge-disjoint branchings

in digraph G(V,E), rooted at node v_0. **Please implement.**