Sparsity: algorithms, approximations, and analysis

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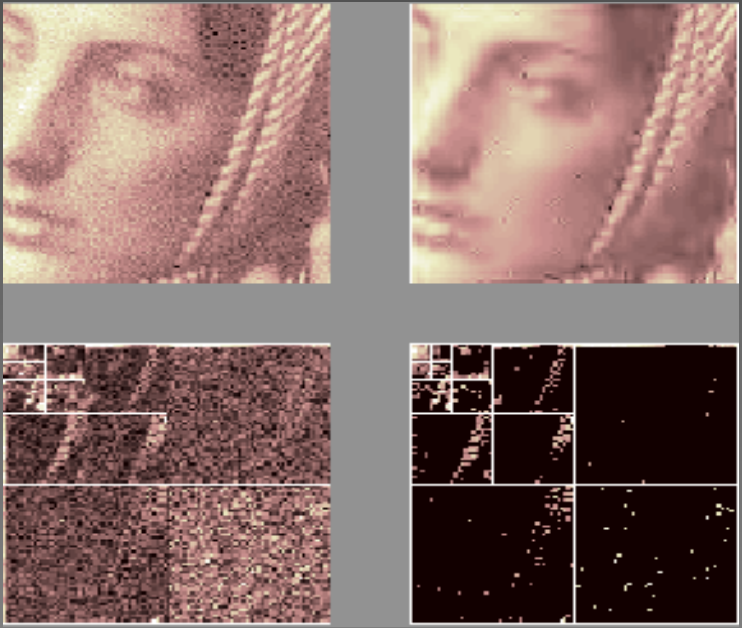
Basic image/signal/data compression: transform coding



Approximate signals sparsely

Compress images, signals, data

accurately (mathematics)

concisely (statistics)

efficiently (algorithms)

Redundancy

If one orthonormal basis is good, surely two (or more) are better...

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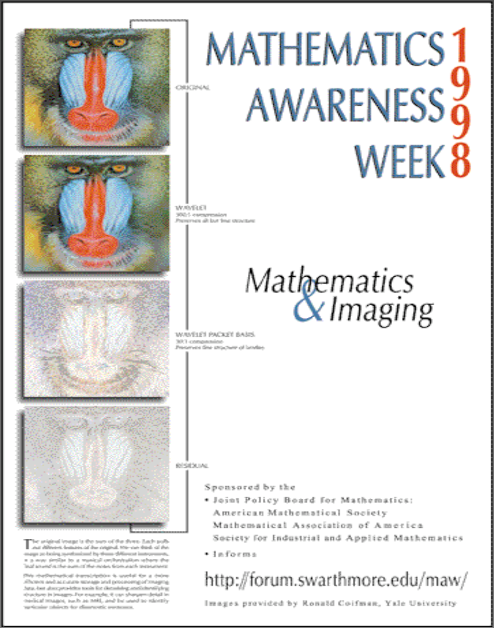
...especially for images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Images provided by Ronald Coifman, Yale University

Dictionary

Definition
A dictionary D in $\mathbb{R}^n$ is a collection $\{\varphi_\ell\}_{\ell=1}^{d} \subset \mathbb{R}^n$ of unit-norm vectors: $\|\varphi_\ell\|_2 = 1$ for all ℓ.

- Elements are called *atoms*
- If $\text{span}\{\varphi_\ell\} = \mathbb{R}^n$, the dictionary is *complete*
- If $\{\varphi_\ell\}$ are linearly dependent, the dictionary is *redundant*

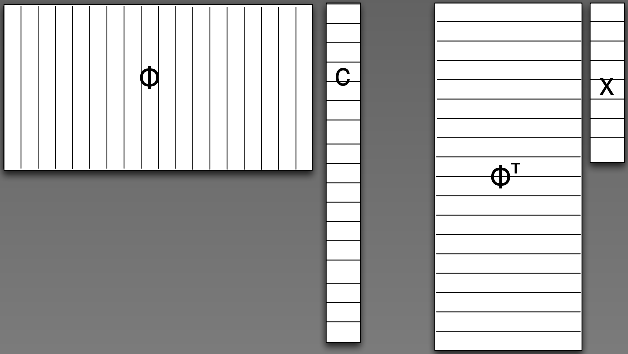
Matrix representation

Form a matrix

$$\Phi = \begin{bmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_d \end{bmatrix}$$

so that

$$\Phi c = \sum_{\ell} c_\ell \varphi_\ell.$$

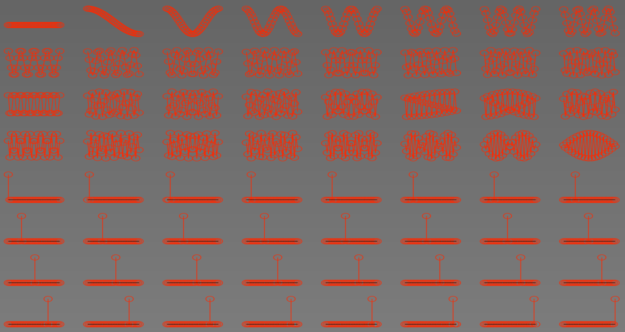


Examples: Fourier—Dirac

$$\Phi = [\,\mathcal{F} \mid I\,]$$

$$\varphi_\ell(t) = \frac{1}{\sqrt{n}} e^{2\pi i \ell t/n} \quad \ell = 1, 2, \ldots, n$$

$$\varphi_\ell(t) = \delta_\ell(t) \quad \ell = n+1, n+2, \ldots, 2n$$



Sparse Problems

Exact. Given a vector $x \in \mathbb{R}^n$ and a complete dictionary Φ, solve

$$\min_c \|c\|_0 \quad \text{s.t.} \quad x = \Phi c$$

i.e., find a sparsest representation of x over Φ.

Error. Given $\epsilon \geq 0$, solve

$$\min_c \|c\|_0 \quad \text{s.t.} \quad \|x - \Phi c\|_2 \leq \epsilon$$

i.e., find a sparsest approximation of x that achieves error ϵ.

Sparse. Given $k \geq 1$, solve

$$\min_c \|x - \Phi c\|_2 \quad \text{s.t.} \quad \|c\|_0 \leq k$$

i.e., find the best approximation of x using k atoms.

NP-hardness

Theorem

Given an arbitrary redundant dictionary Φ and a signal x, it is NP-hard to solve the sparse representation problem $\textsc{Sparse}$.
[Natarajan'95,Davis'97]

Corollary

$\textsc{Error}$ *is NP-hard as well.*

Corollary

It is NP-hard to determine if the optimal error is zero for a given sparsity level k.

Exact Cover by 3-sets: X3C

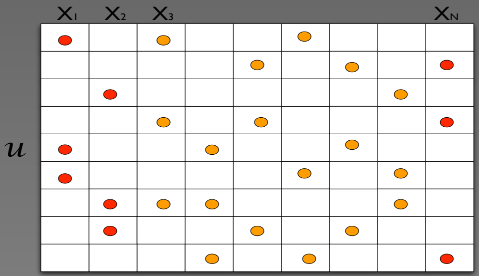
Definition

Given a finite universe $\mathcal{U}$, a collection $\mathcal{X}$ of subsets $X_1, X_2, \ldots, X_d$ s.t. $|X_i| = 3$ for each i, does $\mathcal{X}$ contain a disjoint collection of subsets whose union $= \mathcal{U}$?

Classic NP-hard problem.

Proposition

Any instance of X3C *is reducible in polynomial time to* SPARSE.



Bad news, Good news

Bad news

Given any polynomial time algorithm for SPARSE, there is a dictionary Φ and a signal x such that algorithm returns incorrect answer

Pessimistic: worst case

Cannot hope to approximate solution, either

Bad news, Good news

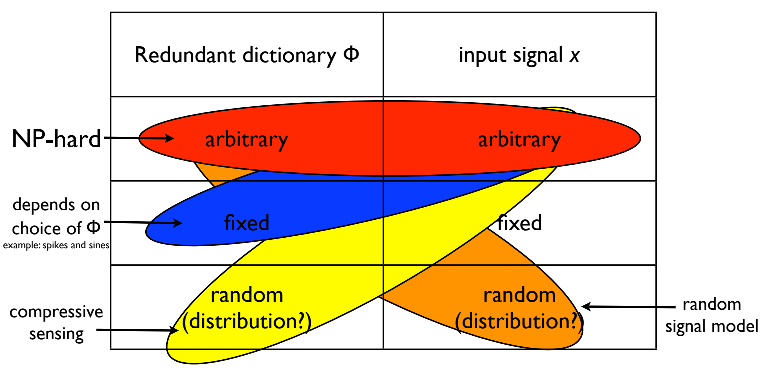
Bad news

- Given any polynomial time algorithm for SPARSE, there is a dictionary Φ and a signal x such that algorithm returns incorrect answer
- Pessimistic: worst case
- Cannot hope to approximate solution, either

Good news

- Natural dictionaries are far from arbitrary
- Perhaps natural dictionaries admit polynomial time algorithms
- Optimistic: rarely see worst case
- Leverage our intuition from orthogonal basis

Hardness depends on instance



Sparse algorithms: exploit geometry

Orthogonal case: pull off atoms one at a time, with dot products in decreasing magnitude

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Orthogonal case: pull off atoms one at a time, with dot products in decreasing magnitude

Why is orthogonal case easy?

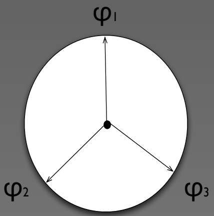
- inner products between atoms are small
- it's easy to tell which one is the best choice

When atoms are (nearly) parallel, can't tell which one is best

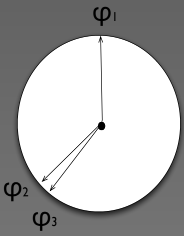
Coherence

Definition
The coherence of a dictionary

$$\mu = \max_{j \neq \ell} \left| \langle \varphi_j, \ \varphi_\ell \rangle \right|$$



Small coherence
(good)



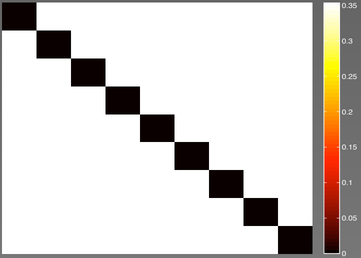
Large coherence
(bad)

Large, incoherent dictionaries

Fourier–Dirac, $d = 2n$, $\mu = \frac{1}{\sqrt{n}}$

wavelet packets, $d = n \log n$, $\mu = \frac{1}{\sqrt{2}}$

There are large dictionaries with coherence close to the lower (Welch) bound; e.g., Kerdock codes, $d = n^2$, $\mu = 1/\sqrt{n}$



Greedy algorithms

Build approximation one step at a time...

...choose the best atom at each step

Orthogonal Matching Pursuit OMP [Mallat'93, Davis'97]

Input. Dictionary Φ, signal x, steps k

Output. Coefficient vector c with k nonzeros, $\Phi c \approx x$

Initialize. counter $t = 1$, $c = 0$

1. **Greedy selection.**

$$\ell_t = \mathrm{argmax} |\Phi^*(x - \Phi c)|$$

2. **Update.** Find $c_{\ell_1}, \ldots, c_{\ell_t}$ to solve

$$\min \left\| x - \sum_s c_{\ell_s} \varphi_{\ell_s} \right\|_2$$

new approximation $a_t \longleftarrow \Phi c$

3. **Iterate.** $t \longleftarrow t + 1$, stop when $t > k$.

Many greedy algorithms with similar outline

Matching Pursuit: replace step 2. by
$c_{\ell_t} \longleftarrow c_{\ell_t} + \langle x - \Phi c,\ \varphi_{\ell_t} \rangle$

Thresholding

Choose m atoms where $|\langle x,\ \varphi_\ell \rangle|$ are among m largest

Alternate stopping rules:

$\|x - \Phi c\|_2 \leq \epsilon$
$\max_\ell |\langle x - \Phi c,\ \varphi_\ell \rangle| \leq \epsilon$

Many other variations

Convergence of OMP

Theorem

Suppose Φ is a complete dictionary for $\mathbb{R}^n$. For any vector x, the residual after t steps of OMP *satisfies*

$$\|x - \Phi c\|_2 \leq \frac{C}{\sqrt{t}}.$$

[Devore-Temlyakov]

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[DEVORE-TEMLYAKOV]

Even if x can be expressed sparsely, OMP may take n steps before the residual is zero.

But, sometimes OMP correctly identifies sparse representations.

Exact Recovery Condition and coherence

Theorem (ERC)

A sufficient condition for OMP *to identify* Λ *after* k *steps is that*

$$\max_{\ell \notin \Lambda} \left\| \Phi_\Lambda^+ \varphi_\ell \right\|_1 < 1$$

where $A^+ = (A^* A)^{-1} A^*$. *[Tropp'04]*

Theorem

The ERC holds whenever $k < \frac{1}{2}(\mu^{-1} + 1)$. *Therefore,* OMP *can recover any sufficiently sparse signals.* *[Tropp'04]*

For most redundant dictionaries, $k < \frac{1}{2}(\sqrt{n} + 1)$.

Sparse representation with OMP

Suppose x has k-sparse representation

$$x = \sum_{\ell \in \Lambda} b_\ell \varphi_\ell \quad \text{where } |\Lambda| = k$$

Sufficient to find Λ—When can OMP do so?

Define

$$\Phi_\Lambda = \begin{bmatrix} \varphi_{\ell_1} & \varphi_{\ell_2} & \cdots & \varphi_{\ell_k} \end{bmatrix}_{\ell_s \in \Lambda} \quad \text{and}$$
$$\Psi_\Lambda = \begin{bmatrix} \varphi_{\ell_1} & \varphi_{\ell_2} & \cdots & \varphi_{\ell_{N-k}} \end{bmatrix}_{\ell_s \notin \Lambda}$$

Define *greedy selection ratio*

$$\rho(r) = \frac{\max_{\ell \notin \Lambda} |\langle r,\ \varphi_\ell \rangle|}{\max_{\ell \in \Lambda} |\langle r,\ \varphi_\ell \rangle|} = \frac{\left\| \Psi_\Lambda^* r \right\|_\infty}{\left\| \Phi_\Lambda^* r \right\|_\infty} = \frac{\text{max i.p. bad atoms}}{\text{max i.p. good atoms}}$$

OMP chooses good atom iff $\rho(r) < 1$

SPARSE

Theorem

Assume $k \le \frac{1}{3\mu}$. For any vector x, the approximation $\widehat{x}$ after k steps of OMP satisfies

$$\|x - \widehat{x}\|_2 \le \sqrt{1+6k}\,\|x - x_k\|_2$$

where x_k is the best k-term approximation to x. [Tropp'04]

Theorem

Assume $4 \le k \le \frac{1}{\sqrt{\mu}}$. Two-phase greedy pursuit produces $\widehat{x}$ s.t.

$$\|x - \widehat{x}\|_2 \le 3\,\|x - x_k\|_2 .$$

Assume $k \le \frac{1}{\mu}$. Two-phase greedy pursuit produces $\widehat{x}$ s.t.

$$\|x - \widehat{x}\|_2 \le \left(1 + \frac{2\mu k^2}{(1-2\mu k)^2}\right)\|x - x_k\|_2 .$$

[Gilbert, Strauss, Muthukrishnan, Tropp'03]

Alternative algorithmic approach

EXACT: non-convex optimization

$$\min \|c\|_0 \quad \text{s.t.} \quad x = \Phi c$$

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ERROR: non-convex optimization

$$\arg\min \|c\|_0 \quad \text{s.t.} \quad \|x - \Phi c\|_2 \leq \epsilon$$

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Convex relaxation of non-convex problem

$$\arg\min \|c\|_1 \quad \text{s.t.} \quad \|x - \Phi c\|_2 \leq \delta.$$

Convex relaxation: algorithmic formulation

Well-studied algorithmic formulation [Donoho, Donoho-Elad-Temlyakov, Tropp, and many others]

Optimization problem = linear program: linear objective function (with variables c^+, c^-) and linear or quadratic constraints

Still need *algorithm* for solving optimization problem

Hard part of analysis: showing solution to convex problem = solution to original problem

Exact Recovery Condition

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Alternate optimization formulations

Constrained minimization:

$$\arg\min \|c\|_1 \quad \text{s.t.} \quad \|x - \Phi c\|_2 \leq \delta.$$

Unconstrained minimization:

$$\text{minimize } L(c; \gamma, x) = \frac{1}{2} \|x - \Phi c\|_2^2 + \gamma \|c\|_1 .$$

Many algorithms for ℓ_1-regularization

Sparse approximation: Optimization vs. Greedy

EXACT and ERROR amenable to convex relaxation and convex optimization

SPARSE not amenable to convex relaxation

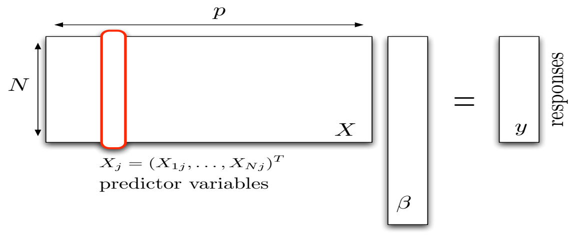
$$\arg\min \|\Phi c - x\|_2 \quad \text{s.t.} \quad \|c\|_0 \leq k$$

but appropriate for greedy algorithms

Connection between...

Sparse Approximation and Statistical Learning

Sparsity in statistical learning



Goal: Given X and y, find α and coeffs. $\beta \in \mathbb{R}^p$ for linear model that minimizes the error

$$(\hat{\alpha}, \hat{\beta}) = \arg\min \|X\beta - (y - \alpha)\|_2^2.$$

Solution:

Least squares: low bias but large variance and hard to interpret lots of non-zero coefficients

Shrink β_j's, make β sparse.

Algorithms in statistical learning

Brute force: Calculate Mallows' C_p for *every* subset of predictor variables, and choose the best one.

Greedy algorithms: Forward selection, forward stagewise, least angle regression (LARS), backward elimination.

Constrained optimization: Quadratic programming problem with linear constraints (e.g., LASSO).

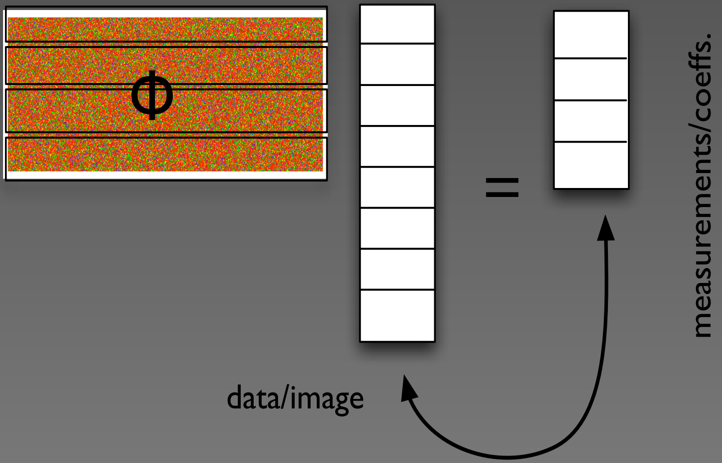
Unconstrained optimization: regularization techniques

Sparse approximation and SVM equivalence [Girosi '96]

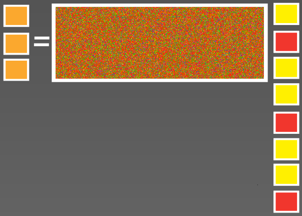
Connection between...

Sparse Approximation and Compressed Sensing

Interchange roles



Problem statement



m as small as possible

Assume x has low complexity: x is k-sparse (with noise)

Construct

Matrix $\Phi \colon \mathbb{R}^n \to \mathbb{R}^m$

Decoding algorithm $\mathcal{D}$

Given Φx for any signal $x \in \mathbb{R}^n$, we can, with high probability, quickly recover $\widehat{x}$ with

$$\|x - \widehat{x}\|_p \leq (1+\epsilon) \min_{y\ k-sparse} \|x - y\|_q = (1+\epsilon)\|x - x_k\|_q$$

Comparison with Sparse Approximation

SPARSE: Given y and Φ, find (sparse) x such that $y = \Phi x$. Return $\widehat{x}$ with guarantee

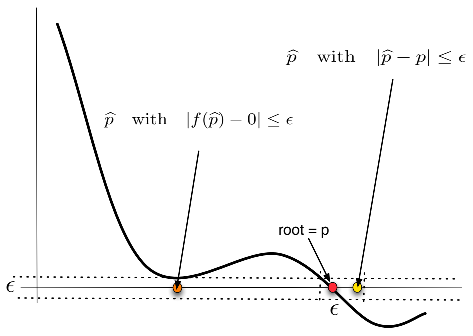
$$\|\Phi \widehat{x} - y\|_2 \quad \text{small compared with } \|y - \Phi x_k\|_2.$$

CS: Given y and Φ, find (sparse) x such that $y = \Phi x$. Return $\widehat{x}$ with guarantee

$$\|\widehat{x} - x\|_p \quad \text{small compared with } \|x - x_k\|_q.$$

p and q not always the same, not always $= 2$.

Analogy: root-finding



SPARSE: Given f (and $y = 0$), find p such that $f(p) = 0$. Return $\widehat{p}$ with guarantee

$$|f(\widehat{p}) - 0| \quad \text{small.}$$

CS: Given f (and $y = 0$), find p such that $f(p) = 0$. Return $\widehat{p}$ with guarantee

$$|\widehat{p} - p| \quad \text{small.}$$

Parameters

Number of measurements m

Recovery time

Approximation guarantee (norms, mixed)

One matrix vs. distribution over matrices

Explicit construction

Universal matrix (for any basis, after measuring)

Tolerance to measurement noise

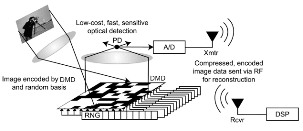
Applications

Data stream algorithms

- x_i = number of items with index i
- can maintain Φx under increments to x
- recover approximation to x

Efficient data sensing

- digital/analog cameras
- analog-to-digital converters
- high throughput biological screening (pooling designs)



Error-correcting codes

- code $\{y \in \mathbb{R}^n | \Phi y = 0\}$
- x = error vector, Φx = syndrome

Two approaches

Geometric [Donoho '04],[Candes-Tao '04, '06],[Candes-Romberg-Tao '05], [Rudelson-Vershynin '06], [Cohen-Dahmen-DeVore '06], and many others...

Dense recovery matrices that satisfy RIP (e.g., Gaussian, Fourier)

Geometric recovery methods (ℓ_1 minimization, LP)

$$\widehat{x} = \mathrm{argmin}\|z\|_1 \text{ s.t. } \Phi z = \Phi x$$

Uniform guarantee: one matrix A that works for all x

Combinatorial [Gilbert-Guha-Indyk-Kotidis-Muthukrishnan-Strauss '02], [Charikar-Chen-FarachColton '02] [Cormode-Muthukrishnan '04], [Gilbert-Strauss-Tropp-Vershynin '06, '07]

Sparse random matrices (typically)

Combinatorial recovery methods or weak, greedy algorithms

Per-instance guarantees, later uniform guarantees

Summary

- Sparse approximation, statistical learning, and compressive sensing intimately related
- Many models of computation and scientific/technological problems in which they all arise
- Algorithms for all similar: optimization and greedy
- Community progress on geometric and statistical models for matrices Φ and signals x, different problem instance types
- Explicit constructions?
- Better/different geometric/statistical models?
- Better connections with coding and complexity theory?