The Kernel Beta Process

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The kernel beta process

**Theorem:** Assume parameters \(\{x_i^*, \psi_i^*, \pi_i, \omega_i\}\) are drawn from measure \(\nu_{\mathcal{X}} = H(dx^*)Q(d\psi^*)\nu(d\pi, d\omega)\), and that the following measure is constituted for any covariate \(x \in \mathcal{X}\):

\[
B_x = \sum_{i=1}^{\infty} \pi_i K(x, x_i^*; \psi_i^*) \delta_{\omega_i}
\]

For any finite set of covariates \(S = \{x_1, \ldots, x_{|S|}\}\), define the random vector \(K = (K(x_1, x^*; \psi^*), \ldots, K(x_{|S|}, x^*; \psi^*))^T\). For \(\forall A \subset \mathcal{F}\), the characteristic function for measures at covariates in \(S\) satisfies

\[
\mathbb{E}[e^{j\langle u, B(A) \rangle}] = \exp\left\{ \int_{\mathcal{X} \times \Psi \times [0,1] \times A} (e^{j\langle u, K\pi \rangle} - 1) \nu_{\mathcal{X}}(dx^*, d\psi^*, d\pi, d\omega) \right\}
\]

with \(\nu_{\mathcal{X}}\) the Lévy measure of the kernel beta process.
Properties of KBP

If \( B \) is drawn from KBP, \( x, x' \in \mathcal{X} \), for \( \forall A \in \mathcal{F} \):

- **Expectation:** 
  \[
  \mathbb{E}[B_x(A)] = B_0(A)\mathbb{E}(K_x)
  \]
  with \( \mathbb{E}(K_x) = \int_{\mathcal{X} \times \psi} K(x, x^*; \psi^*)H(dx^*)Q(d\psi^*) \).

- **Covariance:** 
  \[
  \text{Cov}(B_x(A), B_{x'}(A)) = \mathbb{E}(K_x K_{x'}) \int_{\mathcal{A}} \frac{B_0(d\omega)(1-B_0(d\omega))}{c(\omega)+1} - \text{Cov}(K_x, K_{x'}) \int_{\mathcal{A}} B_0^2(d\omega)
  \]
  (If \( K(x, x^*; \psi^*) = 1 \) for all \( x \in \mathcal{X} \), \( \mathbb{E}(K_x) = \mathbb{E}(K_x K_{x'}) = 1 \), and \( \text{Cov}(K_x, K_{x'}) = 0 \), and the above results reduce to beta process.)

- **Conditional covariance:** 
  With the kernel vectors \( K_x, K_{x'} \) fixed, the conditional covariance is given as:
  \[
  \text{Corr}(B_x(A), B_{x'}(A)) = \frac{\langle K_x, K_{x'} \rangle}{\|K_x\|_2 \cdot \|K_{x'}\|_2}
  \]
Figure: Music temporal correlation: (a) KBP-FA, (b) BP-FA, (c) dHDP-HMM.

Figure: Image denoising result