The Kernel Beta Process

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The kernel beta process

- **Theorem:** Assume parameters $\{x_i^*, \psi_i^*, \pi_i, \omega_i\}$ are drawn from measure $\nu_{\mathcal{X}} = H(dx^*)Q(d\psi^*)\nu(d\pi, d\omega)$, and that the following measure is constituted for *any* covariate $x \in \mathcal{X}$:

$$\mathcal{B}_x = \sum_{i=1}^{\infty} \pi_i K(x, x_i^*; \psi_i^*)\delta_{\omega_i}$$

For any finite set of covariates $\mathcal{S} = \{x_1, \ldots, x_{|\mathcal{S}|}\}$, define the random vector $\mathbf{K} = (K(x_1, x^*; \psi^*), \ldots, K(x_{|\mathcal{S}|}, x^*; \psi^*))^T$. For $\forall \mathcal{A} \subset \mathcal{F}$, the characteristic function for measures at covariates in $\mathcal{S}$ satisfies

$$\mathbb{E}[e^{j<\mathbf{u},\boldsymbol{\mathcal{B}}(\mathcal{A})>}] = \exp\{\int_{\mathcal{X}\times\Psi\times[0,1]\times\mathcal{A}} (e^{j<\mathbf{u},\mathbf{K}\pi>} - 1)\nu_{\mathcal{X}}(dx^*, d\psi^*, d\pi, d\omega)\}$$

with $\nu_{\mathcal{X}}$ the Lévy measure of the kernel beta process.

Properties of KBP

If $\mathcal{B}$ is drawn from KBP, $x, x' \in \mathcal{X}$, for $\forall \mathcal{A} \in \mathcal{F}$:

- **Expectation:** $\mathbb{E}[\mathcal{B}_x(\mathcal{A})] = B_0(\mathcal{A})\mathbb{E}(K_x)$
  with $\mathbb{E}(K_x) = \int_{\mathcal{X}\times\Psi} K(x, x^*; \psi^*)H(dx^*)Q(d\psi^*)$.
- **Covariance:** $\mathrm{Cov}(\mathcal{B}_x(\mathcal{A}), \mathcal{B}_{x'}(\mathcal{A})) =$
  $\mathbb{E}(K_x K_{x'}) \int_{\mathcal{A}} \frac{B_0(d\omega)(1-B_0(d\omega))}{c(\omega)+1} - \mathrm{Cov}(K_x, K_{x'}) \int_{\mathcal{A}} B_0^2(d\omega)$
  (If $K(x, x^*; \psi^*) = 1$ for all $x \in \mathcal{X}$, $\mathbb{E}(K_x) = \mathbb{E}(K_x K_{x'}) = 1$, and $\mathrm{Cov}(K_x, K_{x'}) = 0$, and the above results reduce to beta process.)
- **Conditional covariance:** With the kernel vectors $\mathbf{K}_x, \mathbf{K}_{x'}$ fixed, the conditional covariance is given as:
  $\mathrm{Corr}(\mathcal{B}_x(\mathcal{A}), \mathcal{B}_{x'}(\mathcal{A})) = \frac{<\mathbf{K}_x, \mathbf{K}_{x'}>}{\|\mathbf{K}_x\|_2 \cdot \|\mathbf{K}_{x'}\|_2}$

Experiment - music analysis and image denoising

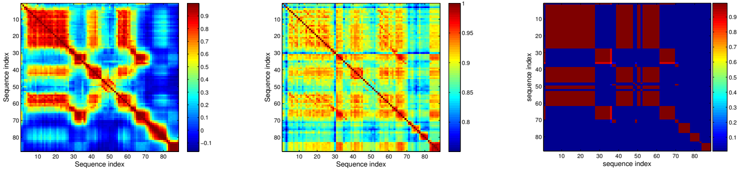

Figure: Music temporal correlation: (a) KBP-FA, (b) BP-FA, (c) dHDP-HMM.

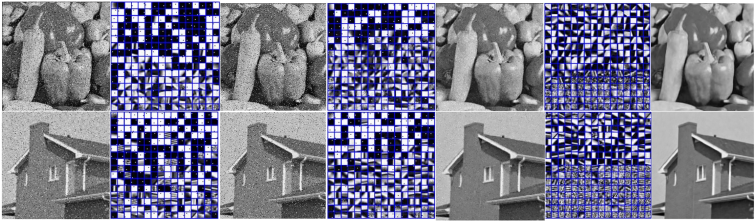

Figure: Image denoising result