An optimal probabilistic graphical model for point set matching

Tiberio Caetano[1,2], Terry Caelli[1] and Dante Barone[2]

[1]Department of Computing Science
University of Alberta, Canada

[2]Instituto de Informática
UFRGS, Brazil

SSPR 2004

Outline

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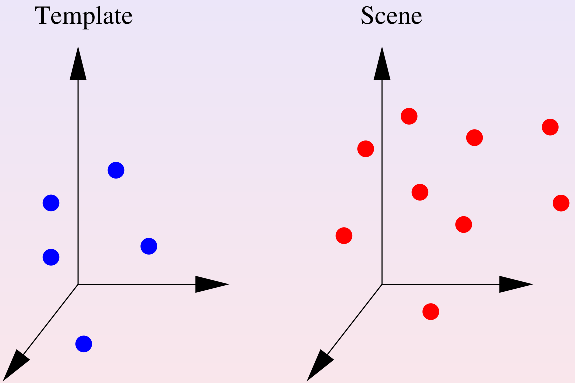
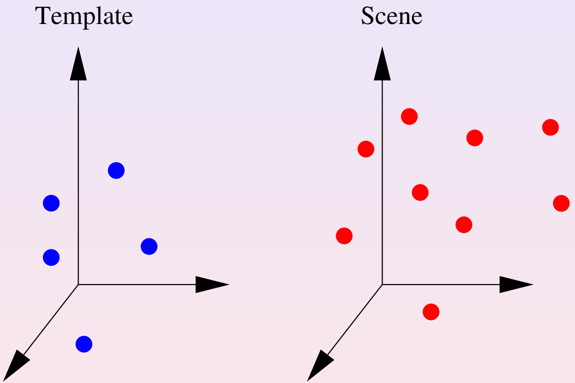
The Problem



f: D –> C

The Problem



f: T –> S

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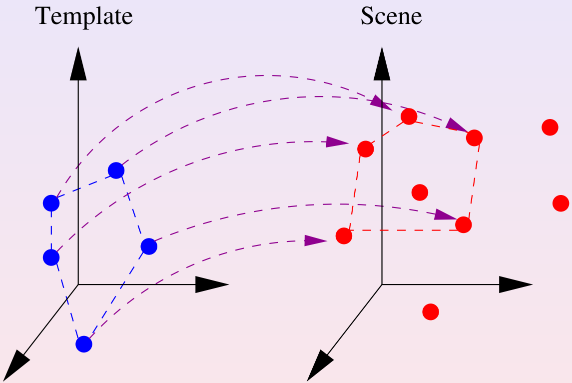

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The Problem as Weighted Graph Matching

EDMs

- The Euclidean Distance Matrix (EDM) of a point set uniquely determines the rigid conformation

As a result...

- Conformations can be compared by comparing the EDMs

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Problem Definition

Problem

Given two point sets $\mathbb{T} = \{d_i, i = 1, \ldots, T\}$ and $\mathbb{S} = \{c_j, j = 1, \ldots, S\}$ in $\mathcal{R}^n$ $(n \in \mathbb{N}^+)$, find the function $f : \mathbb{T} \to \mathbb{S}$ that maximizes

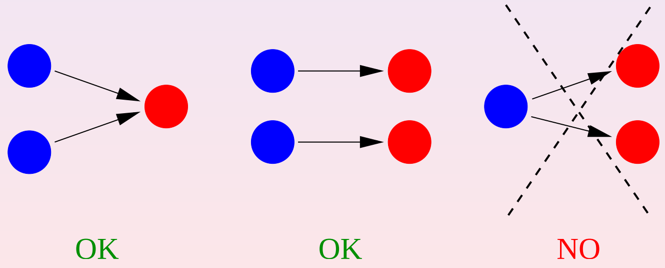
$$P(f) = \sum_{i,j \in \mathbb{T}} \mathcal{S}(||d_i - d_j||, ||c_{f(d_i)} - c_{f(d_j)}||),$$

where $\mathcal{S}(\cdot, \cdot)$ is a similarity function and $|| \cdot ||$ is the L_2 metric

Properties

Properties

- Handles only invariance to isometries (rotations, translations, reflexions)
- f can be *any* function



Computational Complexity

Complexity

- There are S^T possible mapping functions f
- Brute force solution: test each and compute score $P(f)$
- Exponential complexity

Question

How to solve it?

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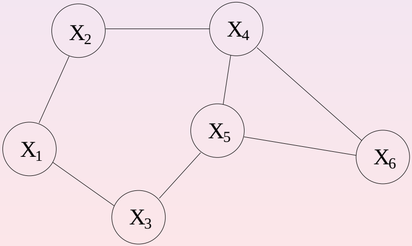
Graphical Models

Graphical Models

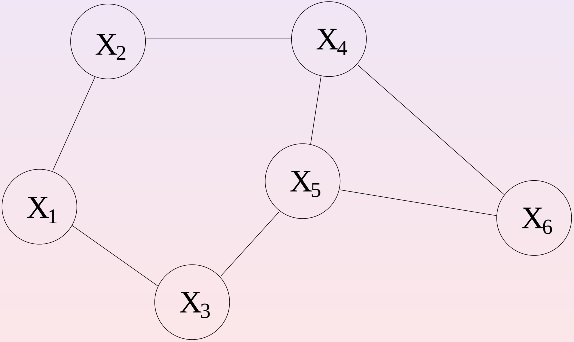
- Graphical Models are stochastic processes defined over graphs
- Nodes of the graph are random variables
- Edges of the graph are probabilistic dependencies between random variables



Types of Graphical Models

Undirected Graphical Models (= Markov Random Fields)

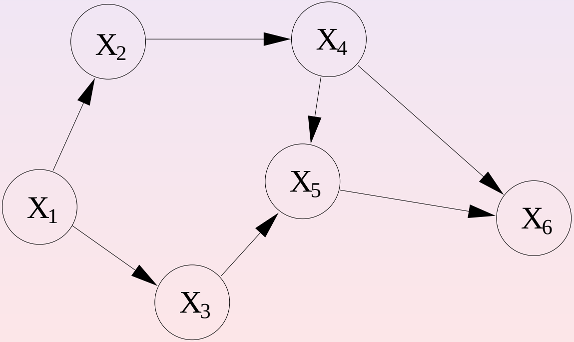
- Symmetric probabilistic relations (undirected edges)



Types of Graphical Models

Directed Graphical Models (= Bayesian Networks)

- Possibly Non-Symmetric probabilistic relations (directed edges)



Features of a Graphical Model

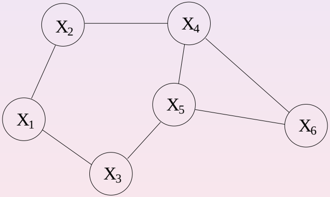
Features

- The probabilistic dependencies between neighbor nodes ("potential functions")
- The connectivity of the graph

Undirected Graphical Models

Undirected

We consider exclusively undirected Graphical Models



Interest

The Interest is to compute the MAP estimate for the model

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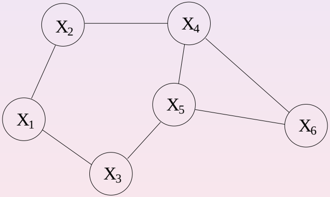

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Hammersley-Clifford Theorem

HC Theorem

States that the joint distribution of a Graphical Model is factorized over products of functions over maximal cliques of the graph:

$$p(x) = \prod_{C \in \mathcal{C}} \psi_C(x_C)$$

Whose maximization is equivalent to minimize

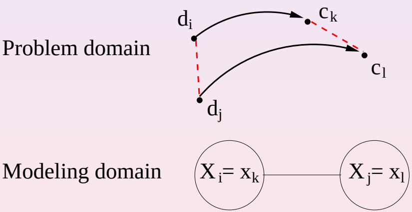
$$U(x) = -\sum_{C \in \mathcal{C}} \log \psi_C(x_C)$$

If we choose only pairwise cliques, then a suitable definition of the $\psi_C s$ leads to the original formulation of the problem as a weighted graph matching one

Our Formulation

Our Formulation

- Template points are random variables
- Scene points are realizations



Thus...

The best mapping becomes the MAP solution of the model.

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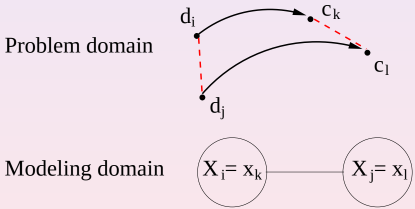


Thus...

The best mapping becomes the MAP solution of the model.

Constructing the model

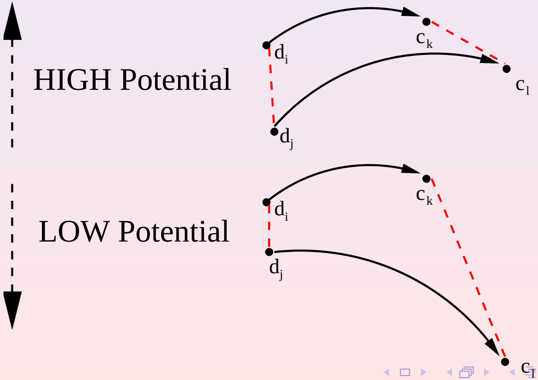
Recall that to construct a Graphical Model one needs

- A set of pairwise potential functions
- A connectivity pattern

Potential Functions

Potential Functions

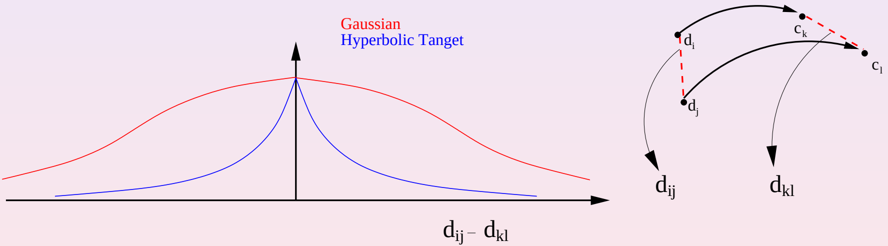
- We use pairwise potential functions
- A potential function measures how good is a pairwise map



Similarity Functions

Possible Similarity Functions

- A potential function ψ is build from a similarity function $\mathcal{S}$ like the following ones:



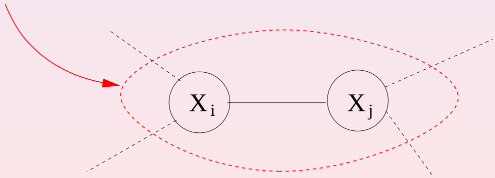
$$\mathbb{G}(d_{ij}, d_{kl}) = \exp\left(-\frac{1}{2\sigma^2}|d_{ij} - d_{kl}|^2\right) \quad \mathbb{H}(d_{ij}, d_{kl}) = 1 - \tanh\left[\frac{|d_{ij} - d_{kl}|}{\sigma}\right]$$

Potential Functions

Potential Functions

- Each connected pair i, j can map to S^2 possible pairs:

$$\psi_{ij}(X_i, X_j) = \frac{1}{Z}\begin{pmatrix} \mathcal{S}(X_i = x_1, X_j = x_1) & \dots & \mathcal{S}(X_i = x_1, X_j = x_S) \\ \vdots & \ddots & \vdots \\ \mathcal{S}(X_i = x_S, X_j = x_1) & \dots & \mathcal{S}(X_i = x_S, X_j = x_S) \end{pmatrix}$$



Connectivity

What is an appropriate connectivity for the graphical model?

- We can't use the fully connected graph because exact inference has exponential complexity

What is an appropriate connectivity for the graphical model?

- It is possible to prove that a particular sparse graph where exact inference is doable in polynomial time is equivalent to the fully connected graph in the limit of exact matching

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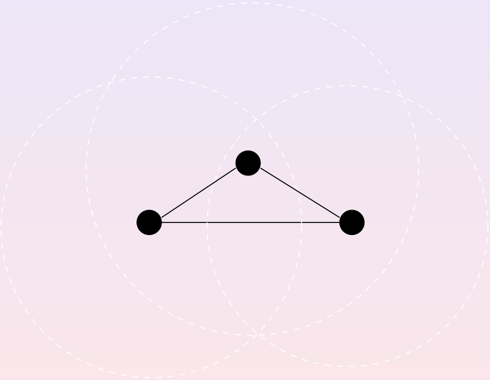

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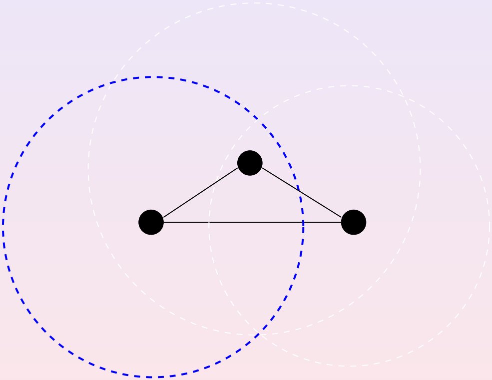

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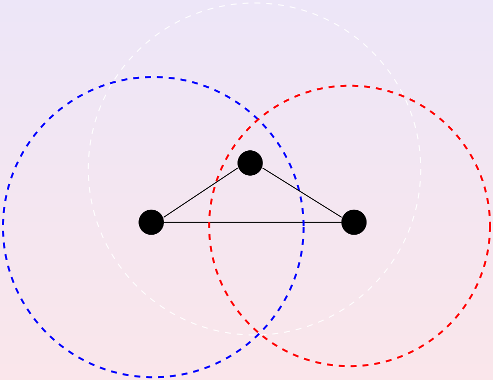

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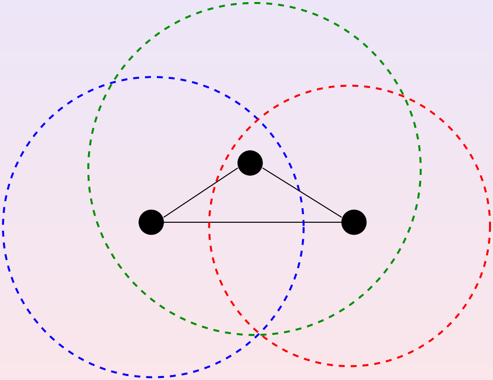

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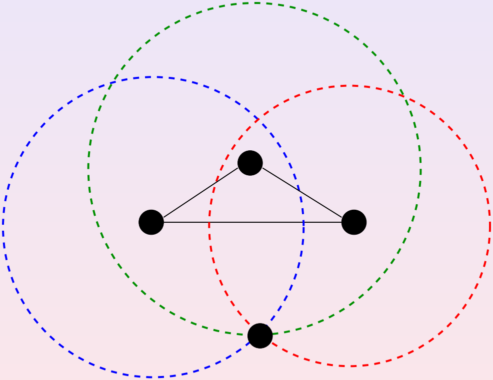

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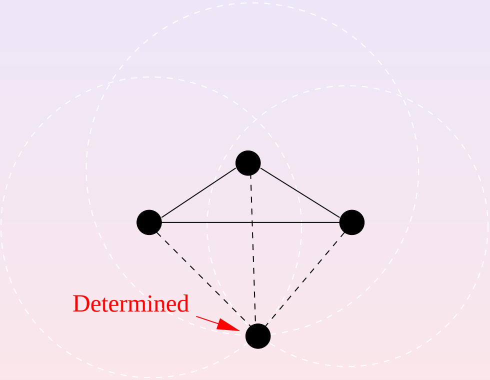

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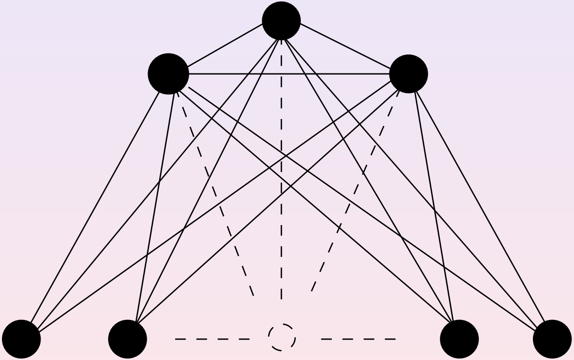

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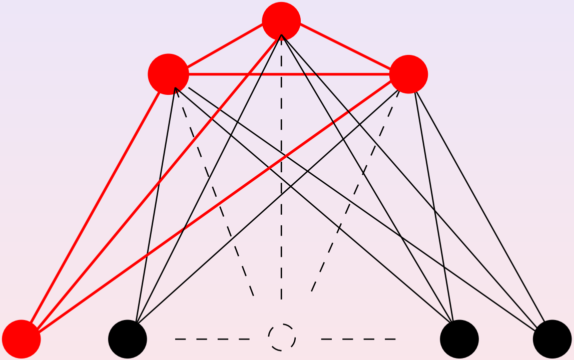

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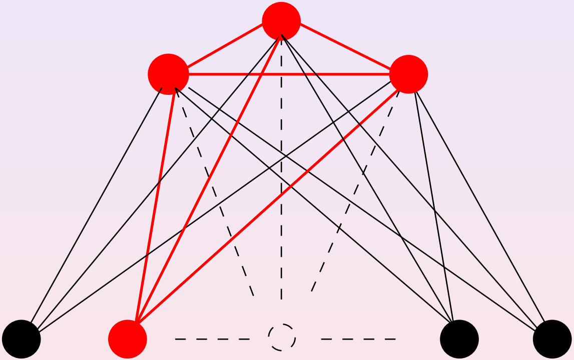

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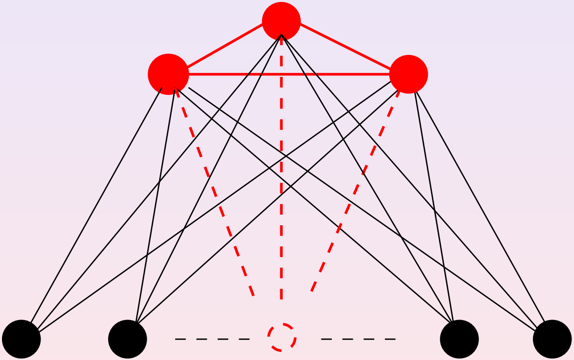

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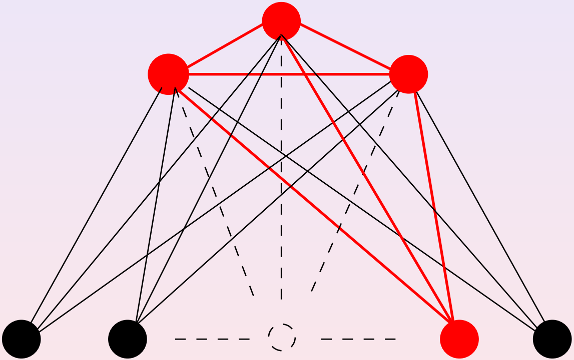

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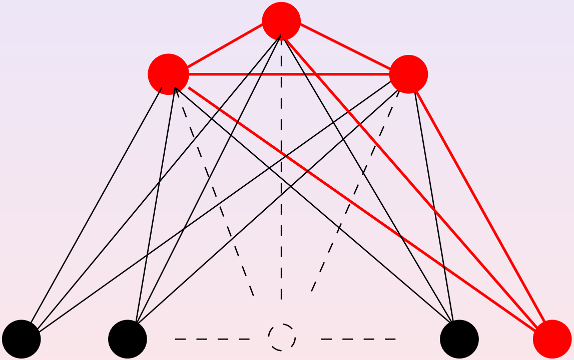

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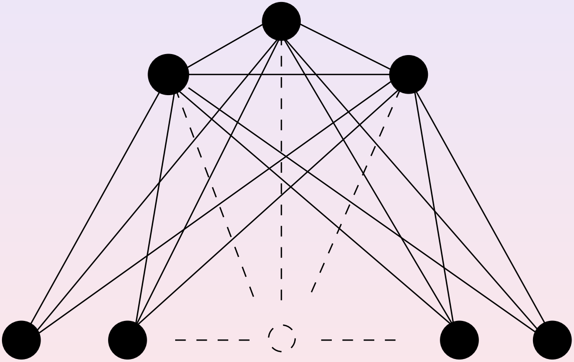
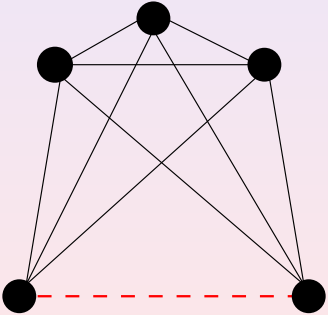


Connectivity

Global Rigidity

- The resulting graph is said to be globally rigid.



R^n

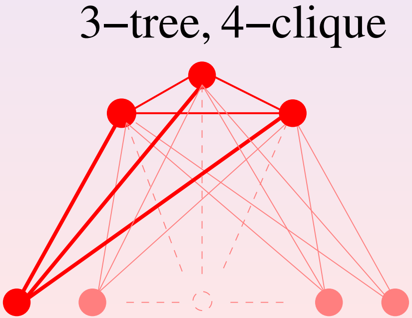
There is a Lemma:

- This can be generalized to any dimension, $n \in \mathcal{N}^+$

Connectivity

k-tree

- The resulting graph is technically a 3-tree, in the case of $\mathbb{R}^2$, and a k-tree in the case of $\mathbb{R}^{k-1}$
- A k-tree has a maximal clique size fixed in k+1



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There is a Theorem:

It is possible to prove that a Graphical Model whose topology is given by a k-tree is equivalent to the fully connected model in the limit of exact matching

Intuition:

The potential functions depend only on the relative distances. Since a k-tree is sufficient to encode all distances, it is also sufficient to encode all potential functions

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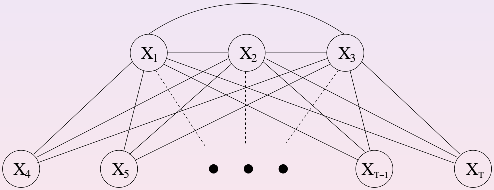

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- It is optimal in the limit of exact matching because it is equivalent to the full model

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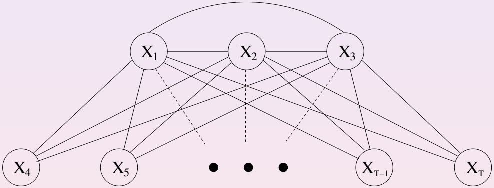

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Optimization

Optimize using JT algorithm (Hugin)

1. Construct hypergraph (a "Junction Tree") where nodes are maximal cliques of original graph
2. "Pass messages" in a systematic way in this hypergraph, what means updating the clique potentials according to specific rules
3. After messages have been passed, the nodes contain the exact MAP estimate for the model

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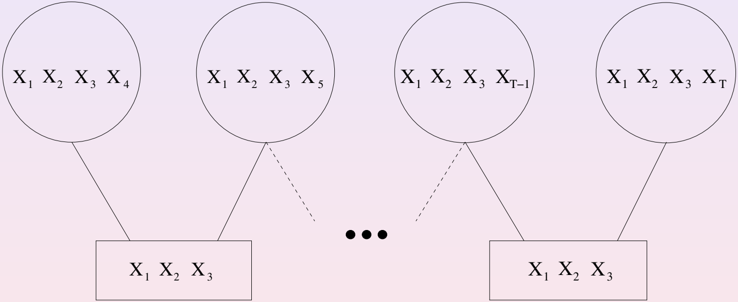

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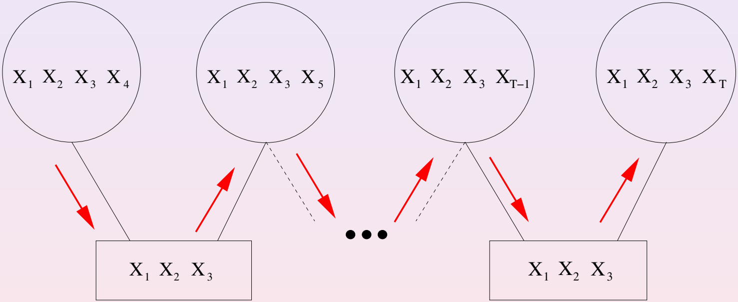

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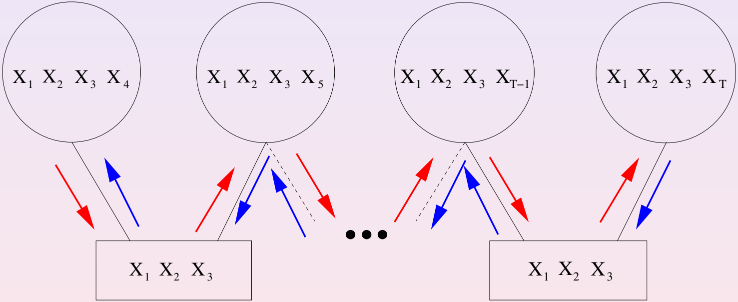

Optimization

Messages

V $\longrightarrow$ W:

$$\phi_S^* = \max_{V \setminus S} \psi_V$$

$$\psi_W^* = \frac{\phi_S^*}{\phi_S} \psi_W$$

Computational Complexity

Complexity

The overall complexity for matching tasks in $\mathbb{R}^{k-1}$ is

$$O(TS^{k+1})$$

T $\longrightarrow$ size of the template pattern
S $\longrightarrow$ size of the scene pattern
k+1 $\longrightarrow$ size of the maximal clique

Outline

Experimental Setup

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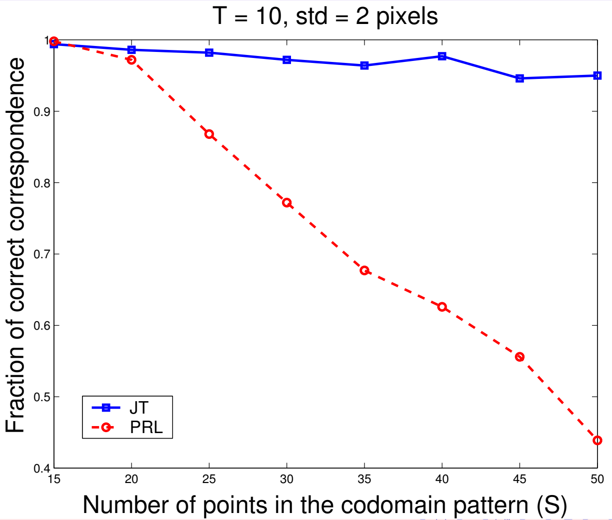
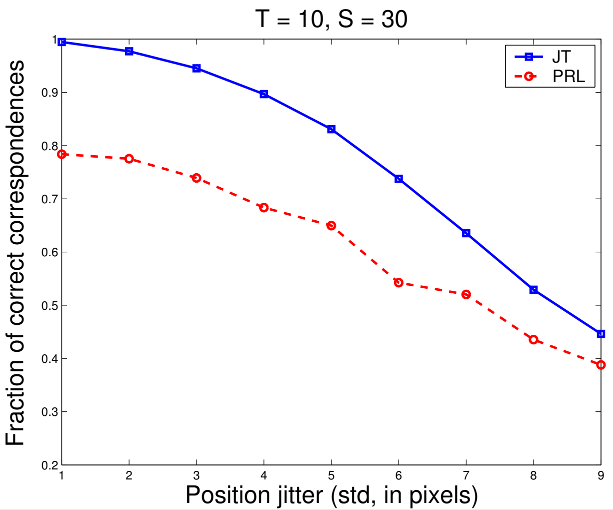
- We compare this approach (JT) with standard Probabilistic Relaxation Labeling (PRL)
- Matching point sets in $\mathbb{R}^2$
- Experiments with inexact matching (for exact matching it always yields perfect results)

Inexact matching: varying problem size



Inexact matching: varying noise



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- Locally optimal
- Iterative

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- Non-iterative (two-pass)

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Point set matching is formulated as inference in a Graphical Model

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Optimal inference is performed in polynomial time in a sparse model which is equivalent to the full model

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The experimental results are perfect for exact matching and significantly better than PRL for inexact matching

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Main Lesson

Redundancies

- What makes search in this problem NP-hard is redundant information
- It can be solved optimally in polynomial time if we properly take advantage of this redundancy

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To do...

Comparison

Compare with other techniques (spectral methods, continuous optimization, etc.)

Invariance

Extend to more complex invariances

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Bounds for error in inexact matching

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Tiberio's Thesis, 2004

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More Info

More information (papers and Thesis)

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