Computing All-Pairs Shortest Paths by Leveraging Low Treewidth

Léon Planken[1] Mathijs de Weerdt[1]
Roman van der Krogt[2]

[1]Delft University of Technology

[2]Cork Constraint Computation Centre

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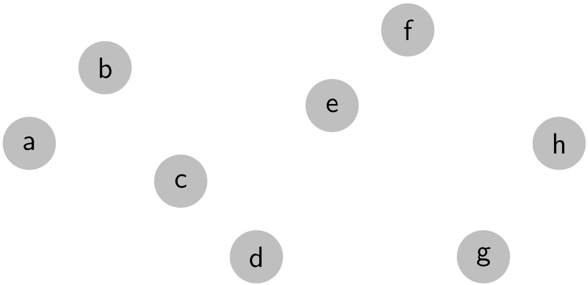
Outline

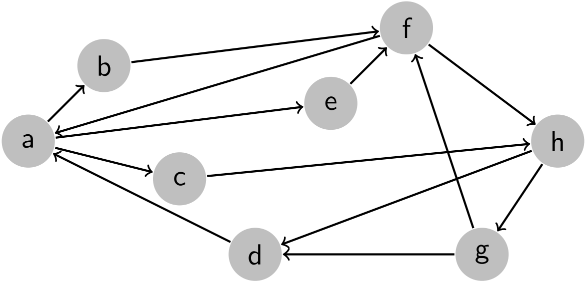
- Computing all-pairs shortest paths
  1. Introduction
  2. Motivation
  3. Existing algorithms
- Leveraging low treewidth

All-pairs shortest paths



We consider directed graphs graphs on n vertices and m edges.

All-pairs shortest paths



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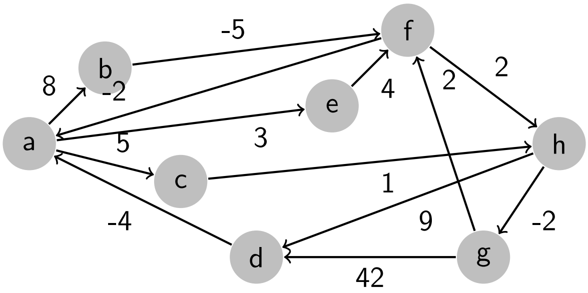
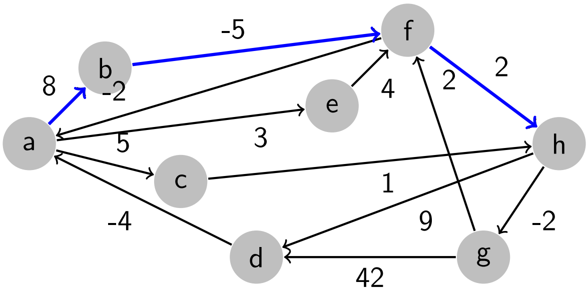


Arcs in the graph are labelled by real-valued weights.

All-pairs shortest paths



We are interested in finding shortest paths. . .

All-pairs shortest paths

D	a	b	c	d	e	f	g	h
a	0	8	5	14	3	3	3	5
b	-7	0	-2	6	-4	-5	-5	-3
c	-1	7	0	10	2	1	-1	1
d	-4	4	1	0	-1	-1	-1	1
e	2	10	7	15	0	4	4	6
f	-2	6	3	11	1	0	0	2
g	0	8	5	13	3	2	0	4
h	-2	6	3	9	1	0	-2	0

... between all pairs of vertices: distance matrix D.

Motivation

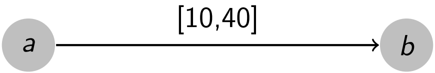
Why is this problem of interest to the ICAPS crowd?

- Shortest paths can clearly be used for spational reasoning
- But for temporal reasoning as well: Simple Temporal Networks

Simple Temporal Networks
Introduction

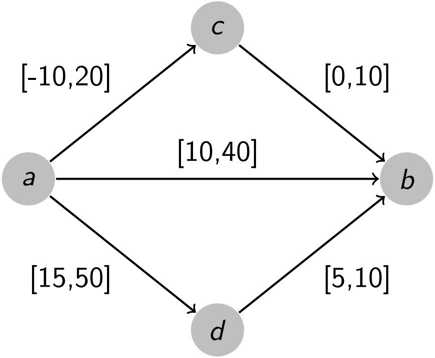
- Proposed in 1991 by Dechter, Meiri and Pearl
- Represent and reason about temporal information
- Nodes represent events
- Arcs represent temporal constraints

Simple Temporal Networks
Example



b happens between 10 and 40 time units after a.

Simple Temporal Networks
Example



Let's add some constraints.

Simple Temporal Networks
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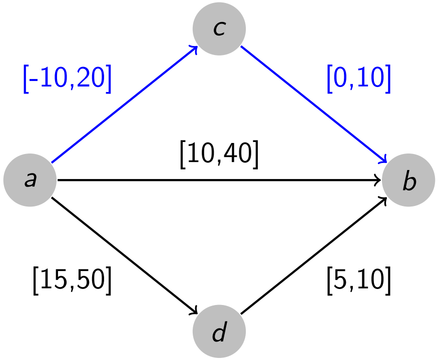

These can be used to infer tighter bounds.

Simple Temporal Networks
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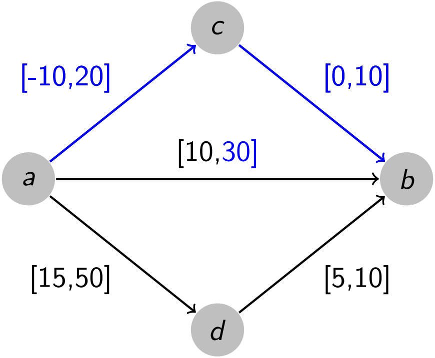

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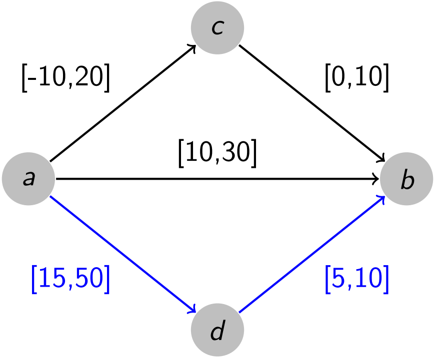

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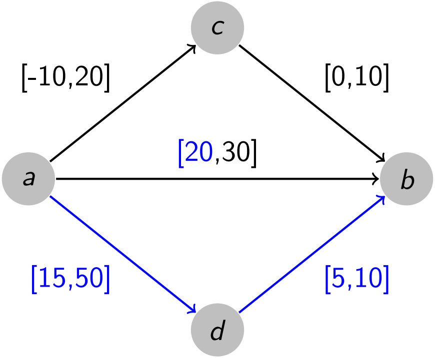

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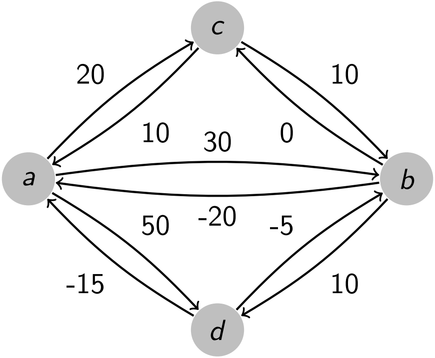

An equivalent representation uses weighted arcs.

Simple Temporal Networks

Applications

- Scheduling problems (e.g. job shop)
- Temporal planning
- Space missions:
  - NASA's Mars Rover
  - ESA's Mars Express

Algorithms for APSP

- Floyd–Warshall (1959–62): $\mathcal{O}\left(n^3\right)$ time
  - Proposed by Dechter et al. for the STN
  - Very simple to implement

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  - Very simple to implement
- Johnson (1977): $\mathcal{O}\left(nm + n^2 \log n\right)$ time
  - Requires Fibonacci heap (1987)
  - A bit harder to implement
  - Benefits from sparseness

Other algorithms

- Bellman–Ford (1958–62): $\mathcal{O}(nm)$ time
  - Find inconsistency (negative cycles)
  - Find single schedule for events
  - Infer constraints involving a single time point

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- Bellman–Ford (1958–62): $\mathcal{O}(nm)$ time
  - Find inconsistency (negative cycles)
  - Find single schedule for events
  - Infer constraints involving a single time point
- $\mathrm{P^3C}$ (ICAPS'08): $\mathcal{O}\left(nw_d^2\right)$ time
  - Infer constraints for an (interesting) subset of all pairs
  - For this specific problem: state of the art

Outline

- Computing all-pairs shortest paths
- Leveraging low treewidth
    1. Directed path consistency
    2. Snowball
    3. Empirical evaluation
    4. Conclusion

Directed path consistency

Introduction

- Proposed by [Dechter et al., 1991] for determining consistency
- Known from CSP literature
- Given vertex ordering d, runs in $\mathcal{O}\left(nw_d^2\right)$ time
- Prerequisite to $\mathrm{P}^3\mathrm{C}$

Directed path consistency
Algorithm

Given $G = \langle V, E \rangle$ and vertex ordering d:

For $k \leftarrow n$ to 1:

For all $i < j < k$ such that $\{i, k\}, \{j, k\} \in E$:

$w_{i\to j} \leftarrow \min\{w_{i\to j},\ w_{i\to k} + w_{k\to j}\}$

$w_{j\to i} \leftarrow \min\{w_{j\to i},\ w_{j\to k} + w_{k\to i}\}$

$E \leftarrow E \cup \{\{i, j\}\}$

If $w_{i\to j} + w_{j\to i} < 0$ return INCONSISTENT

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Directed path consistency
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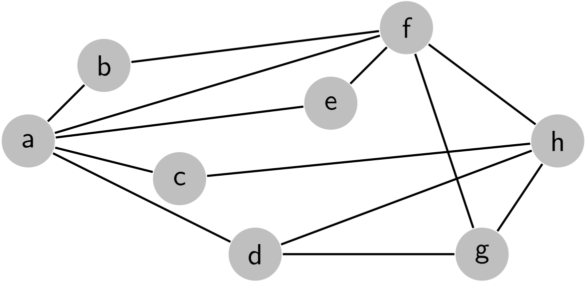

Directed path consistency
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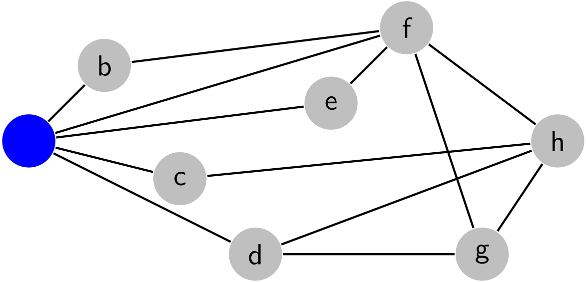

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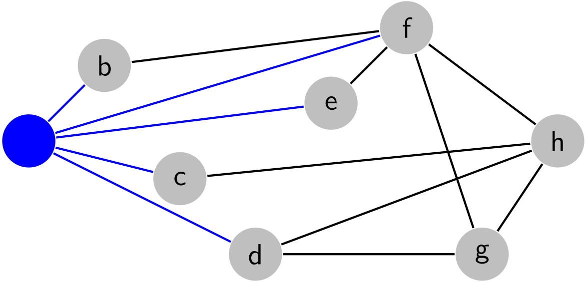

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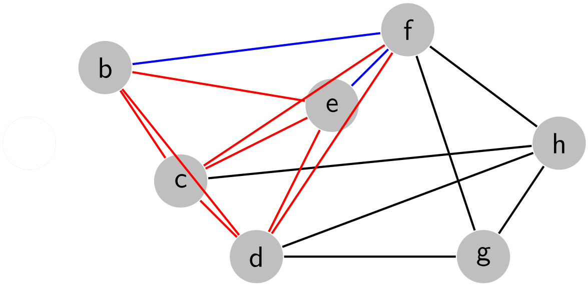

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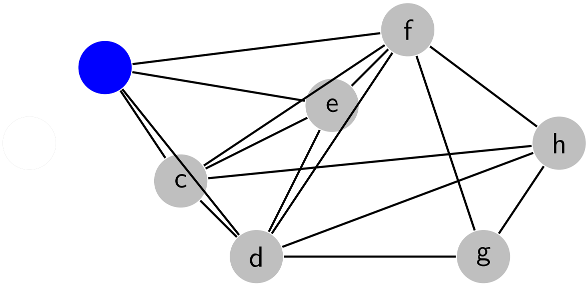

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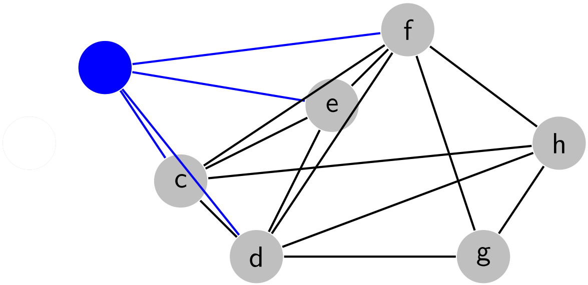

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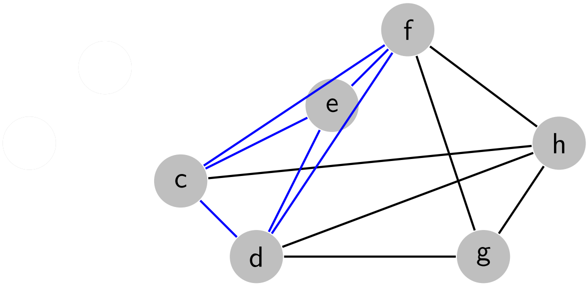

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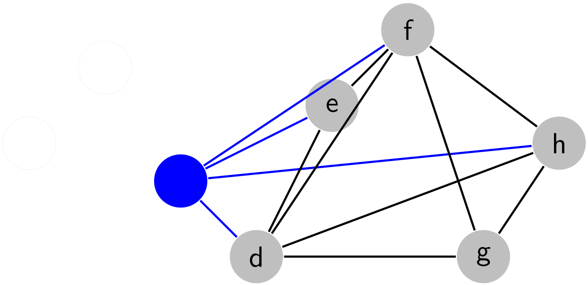

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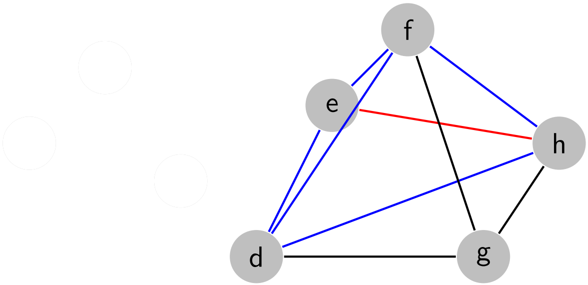

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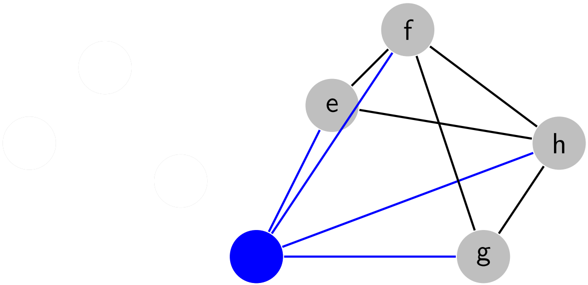

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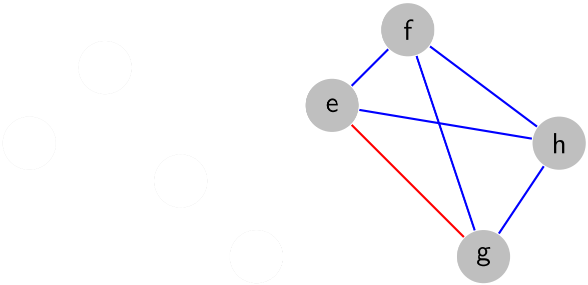

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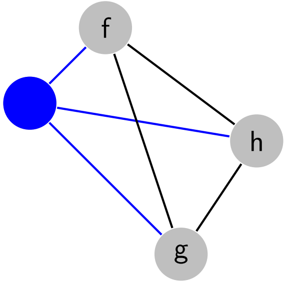

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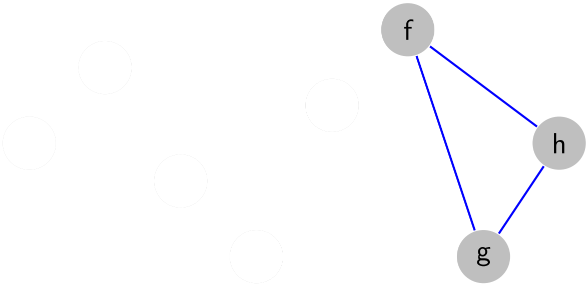

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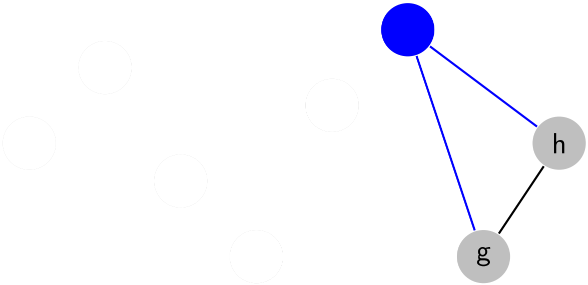


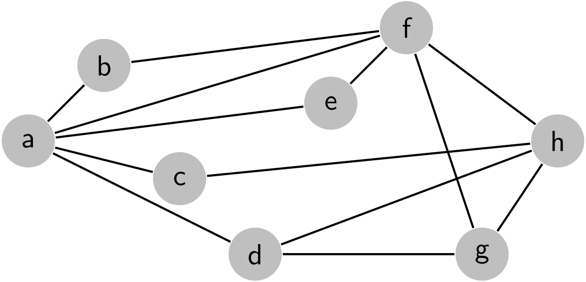
Directed path consistency
Example

Directed path consistency

Example

- Edge between last two vertices is minimal
- This example: $w_d = 5$, $\mathrm{fill} = 9$
- Heuristic: minimum degree

Directed path consistency
Example



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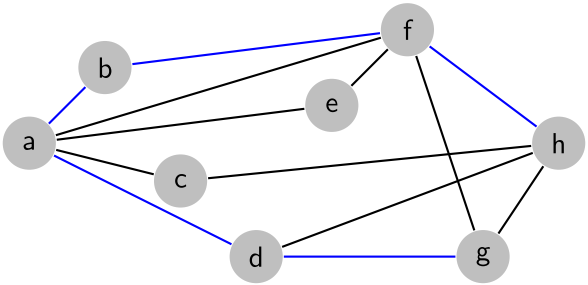

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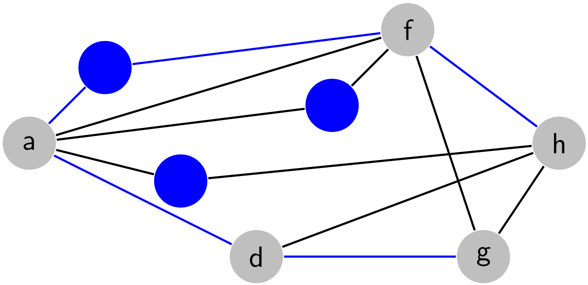

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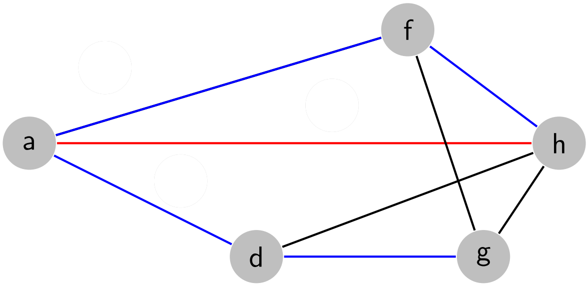

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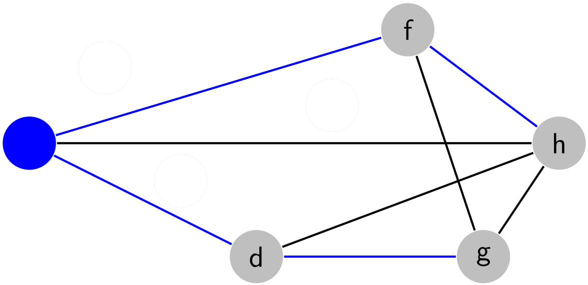

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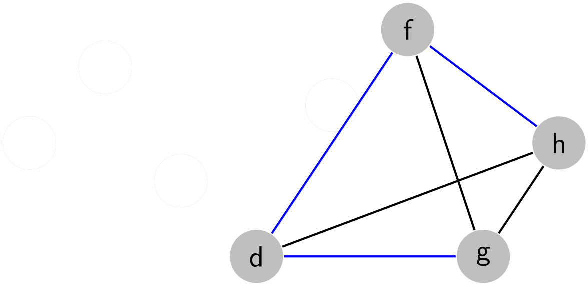

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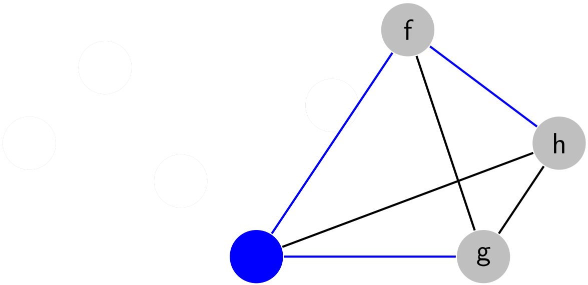

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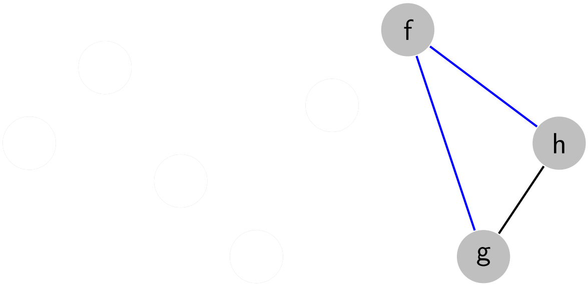

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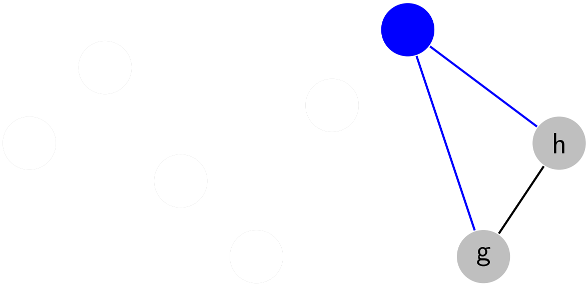

Directed path consistency
Example

Directed path consistency
Example

- Using minimum degree: $w_d = 3$, $\text{fill} = 2$
- Shortest paths can be found by "looking down"
- We use this in our algorithm

Snowball
Introduction

- Like $\mathrm{P}^3\mathrm{C}$, builds on DPC
- Runs in $\mathcal{O}(nm_c) \subseteq \mathcal{O}(n^2 w_d)$
- Idea: after DPC,
  - $D[1][2]$ and $D[2][1]$ are minimal
  - Shortest path to/from k runs through neighbours $j < k$

Snowball
Algorithm

Given $G = \langle V, E \rangle$ which is DPC along d:

For all $i, j \in V : D[i][j] \leftarrow \infty$

For all $i \in V : D[i][i] \leftarrow 0$

For $k \leftarrow 1$ to n:

For all $j < k$ such that $\{j, k\} \in E$:

For $i \in \{1, \ldots, k-1\}$:

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Return D

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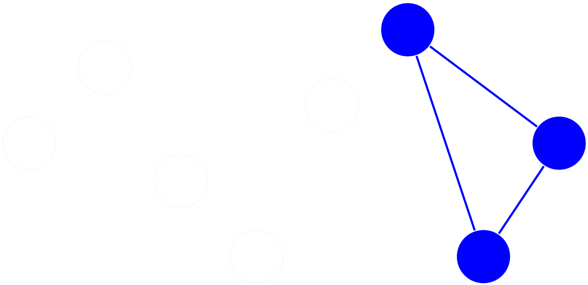

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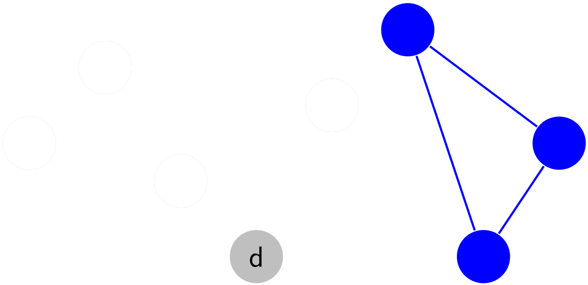

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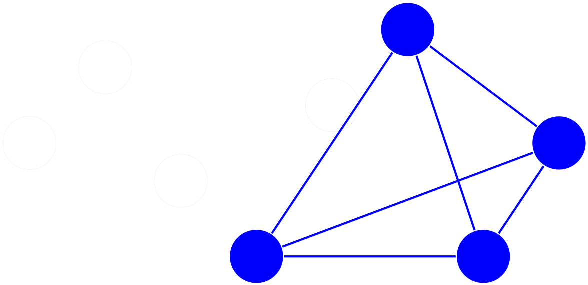

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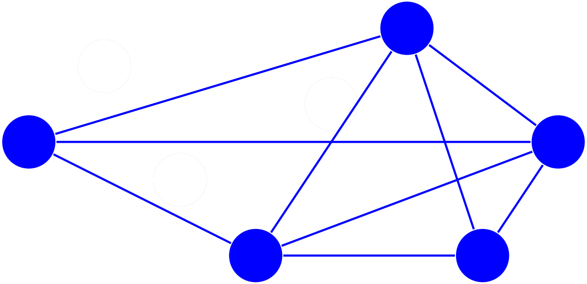

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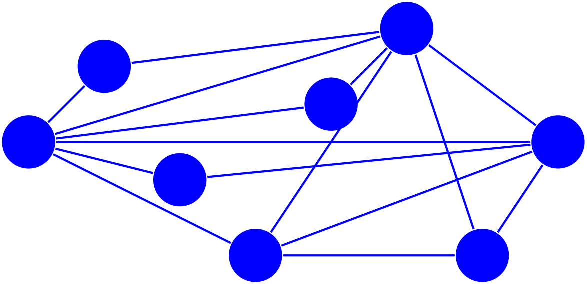


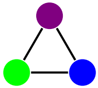
Snowball
Special cases

- If constant w_d exists, it can be found in $\mathcal{O}(n)$ time
  - Then, Snowball runs in $\mathcal{O}(n^2)$ time (optimal).
- Chordal graphs can be identified in $\mathcal{O}(m)$ time
  - Then, Snowball runs in $\mathcal{O}(nm)$ time (also nice).

Snowball
Etymology

Number of computed shortest paths grows quadratically. . .

Snowball
Etymology



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Snowball
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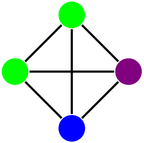

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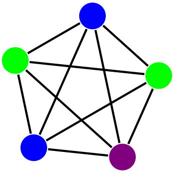

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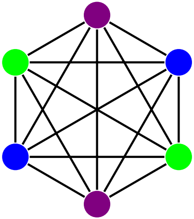

Number of computed shortest paths grows quadratically. . .

Snowball
Etymology

. . . like a snowball.

Empirical evaluation

Introduction

- Compared Floyd–Warshall, Johnson, Snowball
- Java 1.6 (server mode) on Intel Xeon E5430
- Run 10 times, take average CPU time

Empirical evaluation

Benchmark overview

type	#cases	n	m	w_d
Chordal				
- Varying n	130	214–3,125	22,788–637,009	211
- Varying w^*	400	200	985–19,900	5–199
Scale-free				
- Varying n	426	100–200	460–891	38–58
- Varying w_d	190	150	296–2,240	14–103
Diamonds	504	51–379	49–379	2
New York	180	108–4,882	113–8,108	2–51
Job-shop	600	5–241	8–3,840	3–62
HTN	121	500–625	748–1,515	2–144

Empirical evaluation
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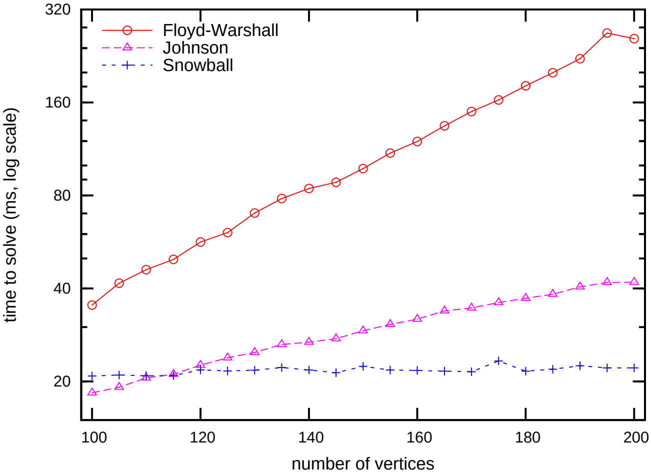
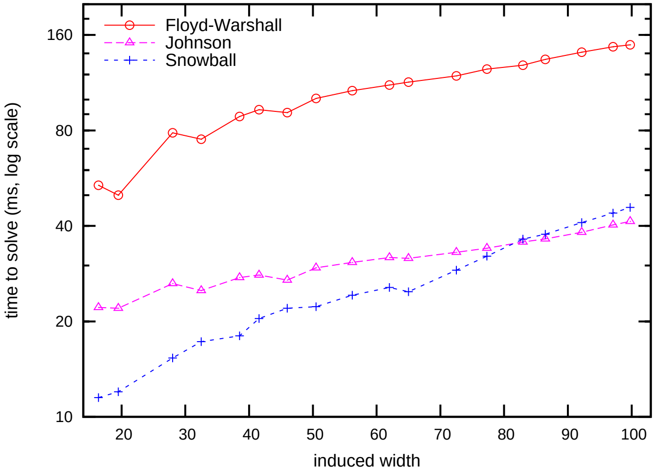
Empirical evaluation

Scale-free graphs, $w_d \in [38, 58]$

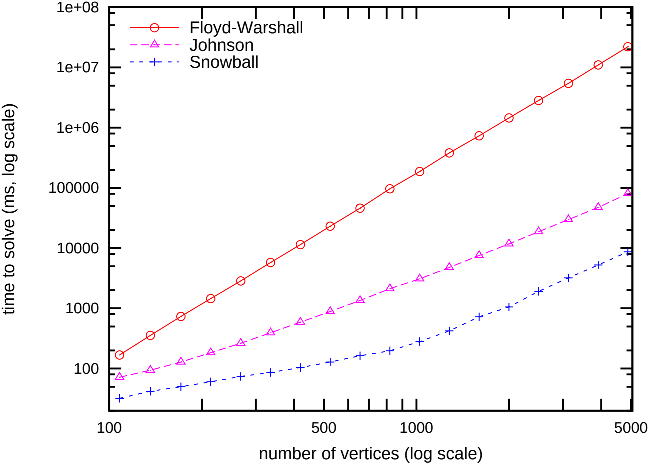


Empirical evaluation

Scale-free graphs, $n = 150$

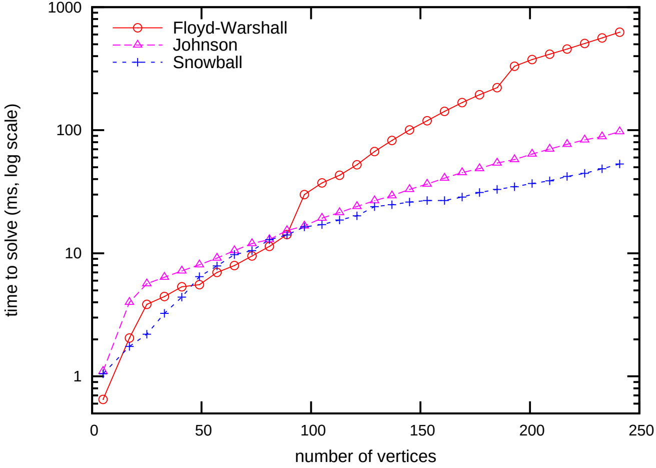


Empirical evaluation
New York City road network, $w_d \in [2, 51]$



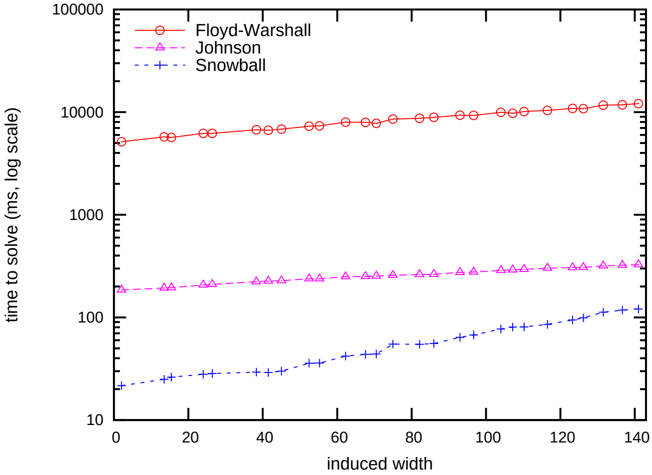
Empirical evaluation

STNs from job-shop benchmarks, $w_d \in [3, 62]$



Empirical evaluation

STNs from HTN benchmarks, $n \in [500, 625]$



Conclusion

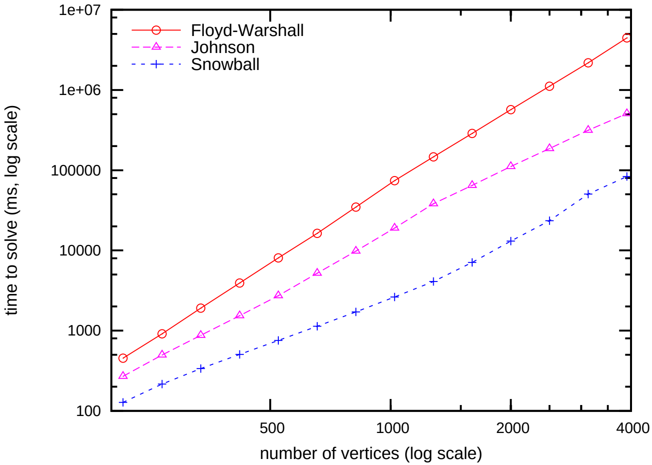
- Proposed a new, simple APSP algorithm:
  - Graphs of constant treewidth: $\mathcal{O}\left(n^2\right)$ time
  - Chordal graphs: $\mathcal{O}\left(nm\right)$ time
  - General graphs: $\mathcal{O}\left(nm_c\right) \subseteq \mathcal{O}\left(n^2 w_d\right)$ time
- Empirically seen to outperform competitors in most cases

Future work

- More efficient Snowball: $\mathcal{O}\left(nw_d^2 + n^2 s\right)$ time, for $s < w_d$
- Compare against [Pettie 2004]: $\mathcal{O}\left(nm + n^2 \log\log n\right)$ time
- Incremental/dynamic version?

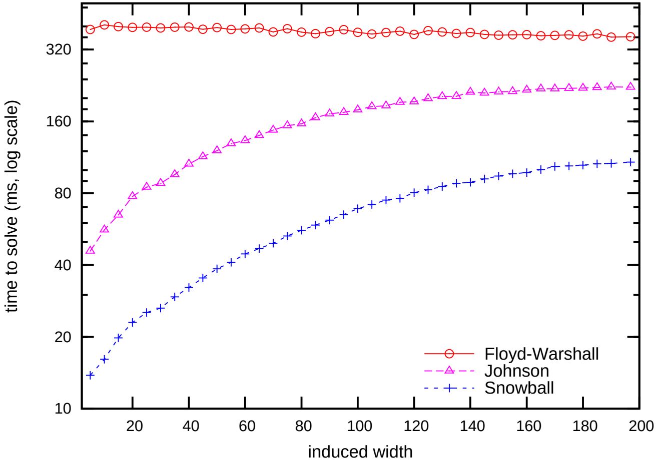
Empirical evaluation

Chordal graphs, $w^* = 211$



Empirical evaluation

Chordal graphs, $n = 200$



Empirical evaluation

"Diamonds" benchmark, $w_d = 2$