MULTI-WAY GAUSSIAN GRAPHICAL MODELS

WITH APPLICATION TO MULTIVARIATE LATTICE DATA

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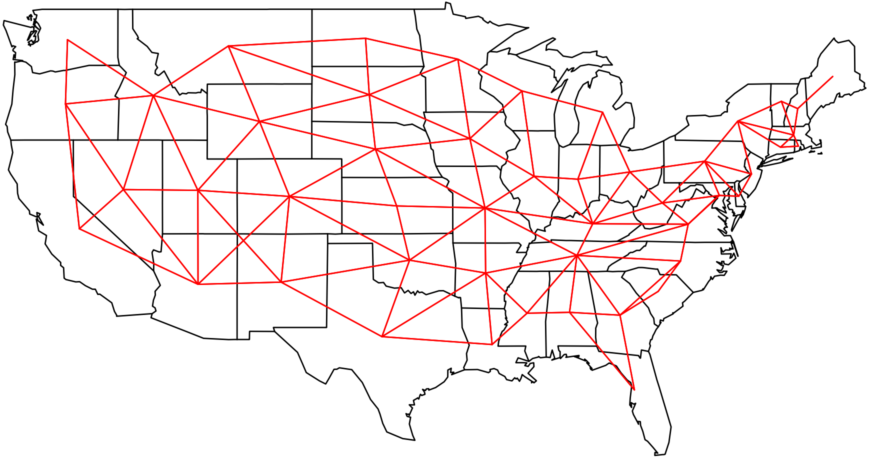
NEIGHBORHOOD GRAPH OF THE STATES OF THE U.S.



Vertices are states. Edges connects states that share a border.

Neighborhood Graph of the States of the U.S.

Maximal prime subgraphs of the U.S. graph

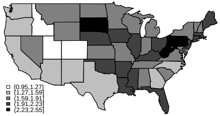
Maximal Prime Subgraph	Members (vertices)
1	Alabama, Florida, Georgia
2	Alabama, Arizona, Arkansas, California, Colorado, Georgia, Idaho, Illinois, Indiana, Iowa, Kansas, Kentucky, Louisiana, Maryland, Michigan, Minnesota, Mississippi, Missouri, Montana, Nebraska, Nevada, New Mexico, North Carolina, North Dakota, Ohio, Oklahoma, Oregon, Pennsylvania, South Dakota, Tennessee, Texas, Utah, Virginia, West Virginia, Wisconsin, Wyoming
3	Delaware, Maryland, Pennsylvania
4	Dist Columbia, Maryland, Virginia
5	Delaware, New Jersey, Pennsylvania
6	New Jersey, New York, Pennsylvania
7	Georgia, North Carolina, South Carolina
8	Idaho, Oregon, Washington
9	Connecticut, Massachusetts, New York
10	Connecticut, Massachusetts, Rhode Island
11	Massachusetts, New York, Vermont
12	Massachusetts, New Hampshire, Vermont
13	Maine, New Hampshire

Example: Cancer Surveillance in the U.S.

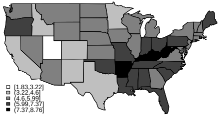
Tiwari et al. (2004), Ghosh & Tiwari (2007), Ghosh et al. (2007, 2008)

- Mortality counts for 11 types of cancers recorded on the 48 continental states plus the District of Columbia for year 2000 based on death certificates collected by the National Center for Health Statistics.
- Source: National Center for Health Statistics; Surveillance, Epidemiology and End Results (SEER) program of the National Cancer Institute.
- Data represented as a 49×11 matrix of counts (geographical region $\times$ cancer).
- Geographical structure is known (neighborhood graph of states).
- Dependencies among cancers are unknown and need to be inferred from the data *in the presence of* the geographical structure.

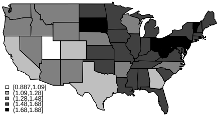
EXAMPLE: CANCER SURVEILLANCE IN THE U.S.



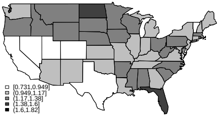
(a) Colon



(b) Lung

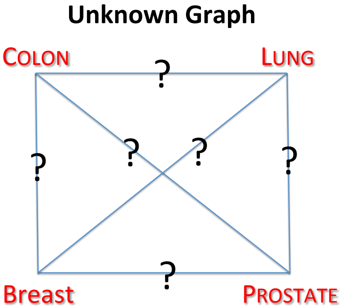


(c) Breast



(d) Prostate

FIGURE: Mortality rates (per 10,000 habitants) in the 48 continental states corresponding to four common cancers during 2000.



Gaussian Graphical Models (GGMs)

Dempster, 1972

- $\mathbf{X} \sim \mathrm{N}_p(\mathbf{0}, \mathbf{K}^{-1})$.
- Graph $G = (V_p, E)$ with vertex set $V_p = \{1, 2, \ldots, p\}$ and edge set $E \subset V_p \times V_p$.

Definition

A Gaussian graphical model with conditional independence graph G is constructed by constraining to zero the off-diagonal elements of $\mathbf{K}$ that do not correspond with edges in G. If $(i, j) \notin E$, X_i and X_j are conditional independent given the remaining variables. The precision matrix $\mathbf{K} = \left(K_{ij}\right)_{1 \leq i,j \leq p}$ is constrained to the cone P_G of symmetric positive definite matrices with off-diagonal entries $K_{ij} = 0$ for all $(i, j) \notin E$.

Conditional Autoregressive (CAR) Models

Cressie, 1973; Besag, 1974; Sun et al., 2000; Gelfand and Vounatsou, 2003

CAR model is defined by the full conditionals:

$$X_i \mid \{X_j : j \neq i\} \sim \mathrm{N}\left(\rho \sum_{j \neq i} \frac{w_{ij}}{w_{i+}} X_j, \frac{\tau^2}{w_{i+}} \right).$$

Then

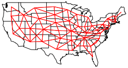
$$\mathbf{X} \sim \mathrm{N}_p\left(\mathbf{0}, \mathbf{D}_W^{-1}(\tau^{-2}, \rho)\right), \; \mathbf{D}_W(\tau^{-2}, \rho) = \tau^{-2}\left(\mathbf{E_W} - \rho \mathbf{W}\right) \in \mathrm{P}_G.$$

- **W** adjacency matrix for graph $G = (V_p, E)$: $w_{ij} = 1 \Leftrightarrow (i, j) \in E$.
- $\mathbf{E_W} = \mathrm{diag}\{w_{1+}, \ldots, w_{p+}\}$.
- ρ (*spatial autocorrelation*).
- $\tau^2 > 0$.

We defined a proper GGM with conditional independence graph G and precision matrix parameterized by ρ and τ^2.

NEIGHBORHOOD STRUCTURE OF THE U.S.

... AS IT APPEARS IN A CONDITIONAL AUTOREGRESSIVE (CAR) MODEL



State	Neighbors
Alabama	Florida, Georgia, Mississippi, Tennessee
Arizona	California, Colorado, Nevada, New Mexico, Utah
Arkansas	Louisiana, Mississippi, Missouri, Oklahoma, Tennessee, Texas
California	Arizona, Nevada, Oregon
Colorado	Arizona, Kansas, Nebraska, New Mexico, Oklahoma, Utah, Wyoming
Connecticut	Massachusetts, New York, Rhode Island
Delaware	Maryland, New Jersey, Pennsylvania
Dist Columbia	Maryland, Virginia
Florida	Alabama, Georgia
Georgia	Alabama, Florida, North Carolina, South Carolina, Tennessee
Idaho	Montana, Nevada, Oregon, Utah, Washington, Wyoming
Illinois	Indiana, Iowa, Kentucky, Missouri, Wisconsin
Indiana	Illinois, Kentucky, Michigan, Ohio
Iowa	Illinois, Minnesota, Missouri, Nebraska, South Dakota, Wisconsin
Kansas	Colorado, Missouri, Nebraska, Oklahoma
Kentucky	Illinois, Indiana, Missouri, Ohio, Tennessee, Virginia, West Virginia
Louisiana	Arkansas, Mississippi, Texas
Maine	New Hampshire
Maryland	Delaware, Dist Columbia, Pennsylvania, Virginia, West Virginia
Massachusetts	Connecticut, New Hampshire, New York, Rhode Island, Vermont
Michigan	Indiana, Ohio, Wisconsin
Minnesota	Iowa, North Dakota, South Dakota, Wisconsin
Mississippi	Alabama, Arkansas, Louisiana, Tennessee
Missouri	Arkansas, Illinois, Iowa, Kansas, Kentucky, Nebraska, Oklahoma, Tennessee
Montana	Idaho, North Dakota, South Dakota, Wyoming
Nebraska	Colorado, Iowa, Kansas, Missouri, South Dakota, Wyoming
Nevada	Arizona, California, Idaho, Oregon, Utah
New Hampshire	Maine, Massachusetts, Vermont
New Jersey	Delaware, New York, Pennsylvania
New Mexico	Arizona, Colorado, Oklahoma, Texas, Utah
New York	Connecticut, Massachusetts, New Jersey, Pennsylvania, Vermont
North Carolina	Georgia, South Carolina, Tennessee, Virginia
North Dakota	Minnesota, Montana, South Dakota
Ohio	Indiana, Kentucky, Michigan, Pennsylvania, West Virginia
Oklahoma	Arkansas, Colorado, Kansas, Missouri, New Mexico, Texas
Oregon	California, Idaho, Nevada, Washington
Pennsylvania	Delaware, Maryland, New Jersey, New York, Ohio, West Virginia
Rhode Island	Connecticut, Massachusetts
South Carolina	Georgia, North Carolina
South Dakota	Iowa, Minnesota, Montana, Nebraska, North Dakota, Wyoming
Tennessee	Alabama, Arkansas, Georgia, Kentucky, Mississippi, Missouri, North Carolina, Virginia
Texas	Arkansas, Louisiana, New Mexico, Oklahoma
Utah	Arizona, Colorado, Idaho, Nevada, New Mexico, Wyoming
Vermont	Massachusetts, New Hampshire, New York
Virginia	Dist Columbia, Kentucky, Maryland, North Carolina, Tennessee, West Virginia
Washington	Idaho, Oregon
West Virginia	Kentucky, Maryland, Ohio, Pennsylvania, Virginia
Wisconsin	Illinois, Iowa, Michigan, Minnesota
Wyoming	Colorado, Idaho, Montana, Nebraska, South Dakota, Utah

The G-Wishart Distribution

Roverato, 2002; Atay-Kayis & Massam, 2005; Letac & Massam, 2007

For $\delta > 2$ and $\mathbf{D}$ positive definite, $\text{Wis}_G(\delta, \mathbf{D})$ is a distribution on the cone P_G with density

$$\Pr(\mathbf{K} \mid G, \delta, \mathbf{D}) = \frac{1}{I_G(\delta, \mathbf{D})} (\det \mathbf{K})^{(\delta-2)/2} \exp\left\{-\frac{1}{2}\langle \mathbf{K}, \mathbf{D}\rangle\right\}.$$

We write $\mathbf{K} \in \text{P}_G$ as

$$\mathbf{K} = Q^T \left(\Psi^T \Psi\right) Q,$$

where Q, Ψ upper triangular and $\mathbf{D}^{-1} = Q^T Q$. The zero constraints on the off-diagonal elements of $\mathbf{K}$ associated with G induce a set of free elements $\Psi^{\nu(G)} = \{\Psi_{ij} : (i,j) \in \nu(G)\}$ of Ψ, where

$$\nu(G) = \{(i,i) : i \in V_p\} \cup \{(i,j) : i < j \text{ and } (i,j) \in E\}.$$

SAMPLING FROM THE G-WISHART DISTRIBUTION

- Block Gibbs Sampler (Piccioni, 2000; Asci & Piccioni, 2007). Requires inversions of large matrices.
- Direct sampler (Carvalho, Massam & West, 2007) works only for decomposable graphs.
- Direct sampler with graph decompositions and accept-reject step (WC) of Wang & Carvalho (2010) works for any graph.
- Metropolis-Hastings algorithm (MME) of Mitsakakis, Massam & Escobar (2011).
- Our Metropolis-Hastings algorithm (DLR): perturb each free element $\Psi^{\nu(G)}$. Advantages: works for any graph, can be used in RJMCMC and can accommodate constraints on elements of **K**, e.g. $K_{11} = 1$.

Table: Monte Carlo estimates of the acceptance probabilities of (DLR), (WC) and (MME) when sampling from $\text{Wis}_{C_p}(103, \mathbf{D}_p)$

p	DLR $\sigma_m = 0.1$	DLR $\sigma_m = 0.5$	DLR $\sigma_m = 1$	DLR $\sigma_m = 2$	WC	MME
4	0.953 (2.0e-3)	0.776 (3.0e-3)	0.600 (3.2e-3)	0.389 (3.8e-3)	0.340 (6.4e-3)	0.473 (1.0e-2)
6	0.947 (2.0e-3)	0.751 (2.4e-3)	0.565 (2.9e-3)	0.356 (2.3e-3)	5.08e-2 (2.2e-3)	0.185 (1.0e-2)
8	0.944 (1.8e-3)	0.740 (2.1e-3)	0.551 (2.3e-3)	0.343 (2.6e-3)	7.94e-3 (7.1e-4)	0.078 (0.9e-2)
10	0.943 (1.7e-3)	0.734 (2.0e-3)	0.543 (2.1e-3)	0.336 (2.1e-3)	1.18e-3 (1.7e-4)	0.035 (8.0e-3)
12	0.942 (1.4e-3)	0.729 (1.8e-3)	0.537 (2.1e-3)	0.331 (2.2e-3)	1.56e-4 (5.1e-5)	0.013 (6.0e-3)
14	0.940 (1.5e-3)	0.725 (1.6e-3)	0.532 (1.8e-3)	0.327 (1.9e-3)	2.27e-5 (1.4e-5)	0.005 (4.0e-3)
16	0.940 (1.3e-3)	0.721 (1.6e-3)	0.527 (1.7e-3)	0.324 (1.7e-3)	2.19e-6 (1.8e-6)	0.002 (3.0e-3)
18	0.939 (1.2e-3)	0.717 (1.4e-3)	0.523 (1.5e-3)	0.321 (1.5e-3)	3.87e-7 (8.0e-7)	0.001 (2.0e-3)
20	0.938 (1.0e-3)	0.714 (1.4e-3)	0.520 (1.6e-3)	0.318 (1.5e-3)	2.08e-8 (3.5e-8)	5.4e-4 (9.0e-4)

CAR Models: Revisited

Classical Approach	New Mixture Approach
$\mathbf{X} \sim \mathrm{N}_p\left(\mathbf{0}, \mathbf{D}_W^{-1}(\tau^{-2}, \rho)\right),$	$\mathbf{X} \sim \mathrm{N}_p\left(\mathbf{0}, \mathbf{K}^{-1}\right),$
	$\mathbf{K} \sim \mathrm{Wis}_G\left(\delta, (\delta-2)\mathbf{D}_W(\tau^{-2}, \rho)\right).$

The G-Wishart prior on **K** induces priors for the regression coefficients in the conditionals:

$$X_i \mid \{X_j : j \neq i\} \quad \sim \quad \mathrm{N}\left(-\sum_{j \in \partial_G(i)} \frac{K_{ij}}{K_{ii}} X_j, \frac{1}{K_{ii}}\right).$$

Greater flexibility w.r.t. the classical parameterization:

$$X_i \mid \{X_j : j \neq i\} \sim \mathrm{N}\left(\rho \sum_{j \in \partial_G(i)} \frac{w_{ij}}{w_{i+}} X_j, \frac{\tau^2}{w_{i+}}\right).$$

Matrix-variate Gaussian Graphical Models

A $p_R \times p_C$ matrix $\mathbf{X}$ that follows a matrix-variate normal distribution

$$\mathrm{vec}(\mathbf{X}) \mid \mathbf{K}_R, \mathbf{K}_C \sim \mathrm{N}_{p_R p_C}\left(\mathbf{0}, [\mathbf{K_C} \otimes \mathbf{K_R}]^{-\mathbf{1}}\right),$$

where

- $\mathbf{K}_R \in \mathrm{P}_{G_R}$, $G_R = \left(V_{p_R}, E_R\right)$ are row precision matrix/graph.
- $\mathbf{K}_C \in \mathrm{P}_{G_C}$, $G_C = \left(V_{p_C}, E_C\right)$ are column precision matrix/graph.

The graphs G_R and G_C define GGMs for the rows and columns of $\mathbf{X}$:

$$\begin{aligned} \mathbf{X}_{i_1 *} \perp\!\!\!\perp \mathbf{X}_{i_2 *} \mid \mathbf{X}_{(V_{p_R} \setminus \{i_1, i_2\})*} & \Leftrightarrow & (\mathbf{K}_R)_{i_1 i_2} = (\mathbf{K}_R)_{i_2 i_1} = 0 \Leftrightarrow (i_1, i_2) \notin E_R, \\ \mathbf{X}_{* j_1} \perp\!\!\!\perp \mathbf{X}_{* j_2} \mid \mathbf{X}_{(V_{p_C} \setminus \{j_1, j_2\})*} & \Leftrightarrow & (\mathbf{K}_C)_{j_1 j_2} = (\mathbf{K}_R)_{j_2 j_1} = 0 \Leftrightarrow (j_1, j_2) \notin E_C. \end{aligned}$$

Since for any $z > 0$,

$$\left(z^{-1}\mathbf{K}_R\right) \otimes (z\mathbf{K}_C) = \mathbf{K}_R \otimes \mathbf{K}_C,$$

need to impose constraint (Wang & West, 2009):

$$(\mathbf{K}_C)_{11} = 1.$$

Multi-way Gaussian Graphical Models

Extension to arrays $\mathbf{X}$ whose elements are indexed by $\{(i_1, i_2, \ldots, i_L) : 1 \leq i_l \leq p_l\}$ and follow an *array normal* distribution

$$\text{vec}(\mathbf{X}) \mid \mathbf{K}_1, \ldots, \mathbf{K}_L \quad \sim \quad \text{N}_{\prod_{l=1}^{L} p_l}(\mathbf{0}, [\mathbf{K}_L \otimes \mathbf{K}_{L-1} \otimes \ldots \otimes \mathbf{K}_1]^{-1}).$$

Here $\mathbf{K}_l \in \text{P}_{G_l}$ defined by a graph G_l associated with dimension l. Independent G-Wishart distributions define a joint prior specification for the separable precision matrix of vec($\mathbf{X}$):

$$\mathbf{K}_1 \mid \delta_1, \mathbf{D}_1 \sim \text{Wis}_{G_1}(\delta_1, \mathbf{D}_1), \; (z_l \mathbf{K}_l) \mid \delta_l, \mathbf{D}_l \quad \sim \quad \text{Wis}_{G_l}(\delta_l, \mathbf{D}_l), \text{ for } l \geq 2.$$

For multivariate lattice data, multi-way GGMs are key in the development of new Bayesian hierarchical models that allow the *simultaneous* use of GGMs to represent *known* spatial dependencies and to determine *unknown* dependencies in the other dimensions of the data (e.g., time).

Multivariate CAR Models

Mardia, 1988; Gelfand & Vounatsou, 2003

Model a matrix $\mathbf{X} = (X_{ij})$, where X_{ij} is the j-th outcome in region i.

$$\begin{aligned}
\mathrm{vec}\left(\widetilde{\mathbf{X}}\right) \mid \boldsymbol{\mu}, \mathbf{K}_R, \mathbf{K}_C &\sim \mathrm{N}_{p_R p_C}\left(\mathrm{vec}\left\{\left(\mathbf{1}_{p_R}\boldsymbol{\mu}^T\right)^T\right\}, [\mathbf{K}_C \otimes \mathbf{K}_R]^{-1}\right), \\
\mathbf{K}_R \mid \delta_R, \mathbf{D}_R &\sim \mathrm{Wis}_{G_R}\left(\delta_R, \mathbf{D}_R\right), \\
(z\mathbf{K}_C) \mid \delta_C, \mathbf{D}_C &\sim \mathrm{Wis}_{G_C}\left(\delta_C, \mathbf{D}_C\right).
\end{aligned}$$

Row graph captures spatial structure, while column graph captures multivariate dependencies across outcomes. Each row vector depends only on its neighbor areas:

$$\mathbf{X}_{i*}^T \mid \left\{\mathbf{X}_{j*}^T : j \neq i\right\} \sim \mathrm{N}_{p_C}\left(-\sum_{j \in \partial_{G_R}(i)} \frac{(\mathbf{K}_R)_{ij}}{(\mathbf{K}_R)_{ii}} \mathbf{X}_{j*}^T, \frac{1}{(\mathbf{K}_R)_{ii}} \mathbf{K}_C^{-1}\right).$$

Example: Cancer Surveillance in the U.S.

Let Y_{ij} be the number of deaths in state $i = 1, \ldots, p_R = 49$ for tumor type $j = 1, \ldots, p_C = 11$. Set

$$\begin{aligned} Y_{ij} \mid \eta_{ij} &\sim \text{Poi}\left(\eta_{ij}\right), \\ \log\left(\eta_{ij}\right) &= \log\left(m_i\right) + \mu_j + X_{ij}, \end{aligned}$$

Here m_i is the population of state i, μ_j is the intercept for tumor type j, and X_{ij} is a zero-mean spatial random effect. Set

$$\delta_R = \delta_C = 3, \mathbf{D}_C = \mathbf{I}_{p_C}, \mathbf{D}_R(\rho) = \mathbf{E}_\mathbf{W} - \rho \mathbf{W},$$
$$\boldsymbol{\mu} \sim \text{N}_{p_C}\left(\boldsymbol{\mu}_0, \boldsymbol{\Omega}^{-1}\right), \boldsymbol{\mu}_0 = \mu_0 \mathbf{1}_{p_C}, \boldsymbol{\Omega} = \omega^{-2} \mathbf{I}_{p_C}.$$

Set μ_0 to the median log incidence rate across all cancers and regions, and ω to be twice the interquartile range in raw log incidence rates.

$$\mathrm{vec}\left(\widetilde{\mathbf{X}}\right) \mid \boldsymbol{\mu}, \mathbf{K}_R, \mathbf{K}_C \quad \sim \quad \mathrm{N}_{p_R p_C}\left(\mathrm{vec}\left\{\left(\mathbf{1}_{p_R}\boldsymbol{\mu}^T\right)^T\right\}, [\mathbf{K}_C \otimes \mathbf{K}_R]^{-1}\right).$$

Model GGM-U

$$\mathbf{K}_R \mid \delta_R, \mathbf{D}_R(\rho) \sim \mathrm{Wis}_{G_R}\left(\delta_R, \mathbf{D}_R(\rho)\right),$$
$$(z\mathbf{K}_C) \mid \delta_C, \mathbf{D}_C \sim \mathrm{Wis}_{G_C}\left(\delta_C, \mathbf{D}_C\right),$$

uniform prior $\Pr(G_C) = 2^{-m}$,

$$m = \binom{p_C}{2}$$

Model GGM-S

$$\mathbf{K}_R \mid \delta_R, \mathbf{D}_R(\rho) \sim \mathrm{Wis}_{G_R}\left(\delta_R, \mathbf{D}_R(\rho)\right),$$
$$(z\mathbf{K}_C) \mid \delta_C, \mathbf{D}_C \sim \mathrm{Wis}_{G_C}\left(\delta_C, \mathbf{D}_C\right),$$

$\Pr(G_C) = \frac{1}{m+1}\binom{m}{\#(G_C)}^{-1}$, *size based prior*

Model FULL

$$\mathbf{K}_R \mid \delta_R, \mathbf{D}_R(\rho) \sim \mathrm{Wis}_{G_R}\left(\delta_R, \mathbf{D}_R(\rho)\right),$$
$$(z\mathbf{K}_C) \mid \delta_C, \mathbf{D}_C \sim \mathrm{Wis}_{G_C}\left(\delta_C, \mathbf{D}_C\right),$$
$$G_C = \text{complete graph}$$

Model MCAR *Gelfand & Vounatsou (2003)*

$$\mathbf{K}_R = \mathbf{D}_R(\rho),$$
$$\mathbf{K}_C \mid \delta_C, \mathbf{D}_C \sim \mathrm{Wis}_{p_C}\left(\delta_C, \mathbf{D}_C\right),$$

$$\rho \sim \mathrm{Uni}(\{0, 0.05, 0.1, \ldots, 0.8, 0.82, \ldots, 0.90, 0.91, \ldots, 0.99\}).$$

Table: Ten-fold cross-validation predictive scores.

Model	*MSE*	*VAR*	*RPS*
GGM-U	17379.9	**23685.2**	62.1
GGM-S	**16979.8**	24361.1	**61.8**
FULL	18959.6	24530.4	63.2
MCAR	19211.1	47568.7	76.7

$\textbf{MSE}(\mathcal{M}) \propto \sum_{\{(i,j):Y_{ij}\geq 25\}} \left(\mathrm{E}_{\mathcal{M}}[Y_{ij}^{cv}(\mathcal{M})] - Y_{ij}\right)^2$, $\textbf{VAR}(\mathcal{M}) \propto \sum_{\{(i,j):Y_{ij}\geq 25\}} \mathrm{Var}_{\mathcal{M}}[Y_{ij}^{cv}(\mathcal{M})]$ (Gelfand and Ghosh, 1998)

RPS = ranked probability score (Czado, Gneiting & Held, 2009)

EXAMPLE: CANCER SURVEILLANCE IN THE U.S.

COMPARISON OF THE PREDICTIVE PERFORMANCE OF THE FOUR MODELS

TABLE: Nominal coverage rates and mean length of the in-sample 95% credible intervals.

Model	Coverage rate	Mean length
GGM-U	0.960	**65.464**
GGM-S	0.964	65.474
FULL	0.964	65.885
MCAR	0.990	69.239

EXAMPLE: CANCER SURVEILLANCE IN THE U.S.

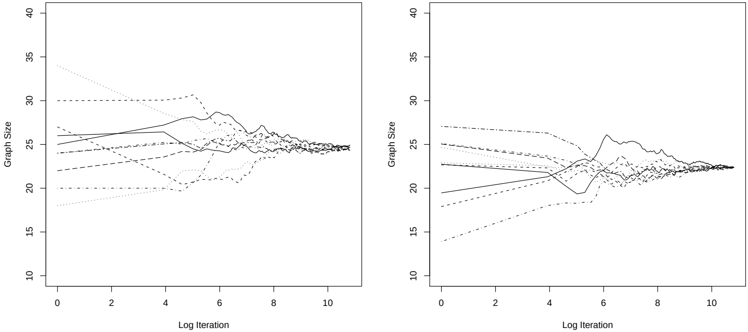


FIGURE: Convergence plot of the average size of the column graph G_C by log iteration for model GGM-U (left panel) and model GGM-S (right panel). The average graph size under both priors is 27.5

Example: Cancer Surveillance in the U.S.

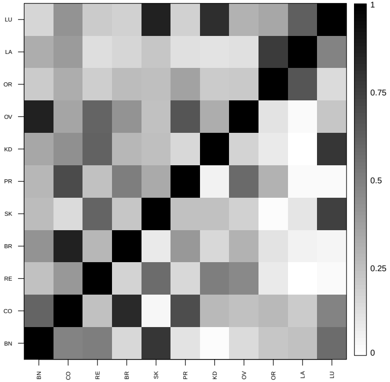


FIGURE: Edge inclusion probabilities for model GGM-U (lower triangle) and model GGM-S (upper triangle) in the U.S. cancer mortality example. The 11 cancers are: brain (BN), breast (BR), colon (CO), kidney (KD), larynx (LA), lung (LU), oral (OR), ovarian (OV), prostrate (PR), skin (SK), and rectum (RE).

EXAMPLE: CANCER SURVEILLANCE IN THE U.S.

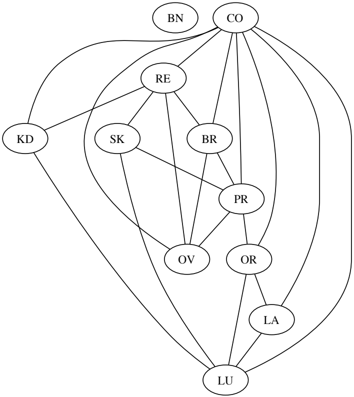


FIGURE: Median posterior column graph from model GGM-U. The 11 cancers are: brain (BN), breast (BR), colon (CO), kidney (KD), larynx (LA), lung (LU), oral (OR), ovarian (OV), prostrate (PR), skin (SK), and rectum (RE).

Example: Low Birth Weight in North Carolina Infants

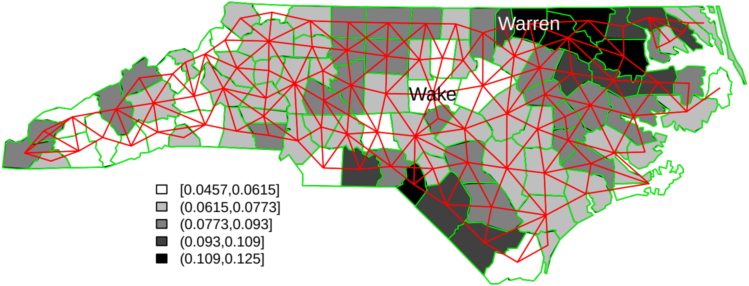
Possible factors that lead to low birth weight ($\leq$ 2500 grams) infants (LBW, yes/no):

1. gender (G, male/female),
2. mother's age at birth dichotomized as younger or older than 35 (MAGE, yes/no),
3. race (R, white non-hispanic/other).

Data collected by the North Carolina State Center for Health Statistics that are available from the North Carolina Vital Statistics Dataverse (`http://arc.irss.unc.edu/dvn/dv/NCVITAL`). The births recorded in 2006, 2007 and 2008 are grouped by the 100 counties in which the mothers reside.

EXAMPLE: LOW BIRTH WEIGHT IN NORTH CAROLINA INFANTS

FIGURE: Proportion of low birthweight infants in the 100 counties of North Carolina. The red lines show the neighborhood structure of the counties as an undirected graph.



Example: Low Birth Weight in North Carolina Infants

- Number of recorded births in Wake county: 37935.
- Number of recorded births in Warren county: 626.

Wake county		Warren county	
Posterior prob.	Model	Posterior prob.	Model
0.9825	[LBW,R][MAGE,R][G,LBW]	0.4581	[R][LBW][MAGE][G]
0.0107	[LBW,R][MAGE,R][G]	0.2498	[LBW,R][MAGE][G]
0.0065	[MAGE,LBW,R][G,LBW]	0.0773	[R][MAGE,LBW][G]
0.0001	[LBW,R][MAGE,R][G,MAGE]	0.0422	[LBW,R][MAGE,LBW][G]
0.0001	[MAGE,LBW,R][G]	0.0334	[R][MAGE][G,LBW]

Table: Five most probable decomposable graphical log-linear models for Wake and Warren counties.

Example: Understanding Trends in Exchange Rate Fluctuations

Modeling Temporal Dependencies

Data: 1000 daily returns on 11 currencies relative to the United States dollar between November 1993 and August 1996.

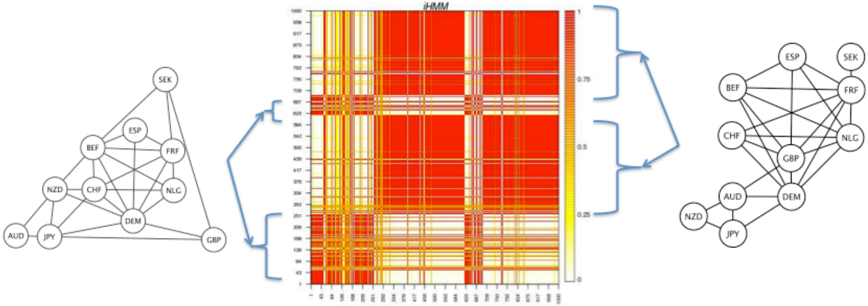


FIGURE: Graphical models associated with the first regime (left) and the second regime (right) in the exchange rate example. These graphs were constructed by adding any edge that had greater than 80% posterior inclusion probability for the respective timepoint. The heatmap (center) displays the probability that two observations belong to the same regime.

PAPERS AND SOFTWARE

... are available from my webpage:

http://www.stat.washington.edu/adobra/