Topological data analysis

Between topology and computer science

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- designing *algorithms* for *computing topological invariants* like homology and cohomology groups, cohomology rings, Steenrod operations, spectral sequences,

closely connected to the better established field of *computational geometry*.

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- This enables the use of a wide topological “arsenal”, for example homology, homotopy, topological simplifications.

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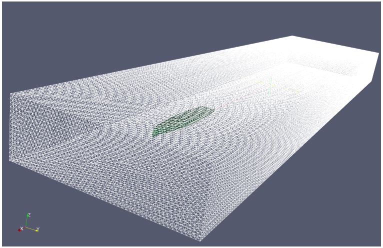

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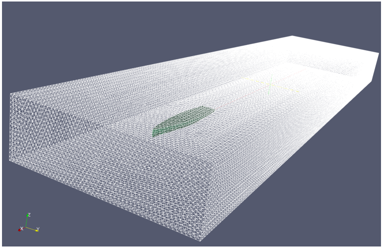

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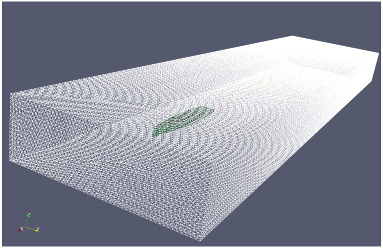

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- and efficient algorithms.

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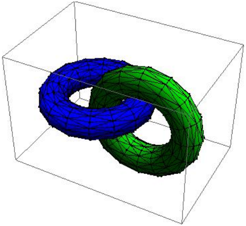
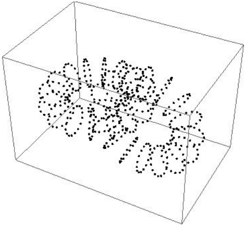
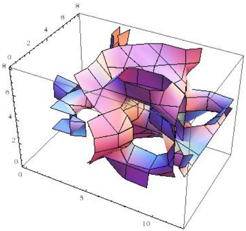


sample



reconstruction



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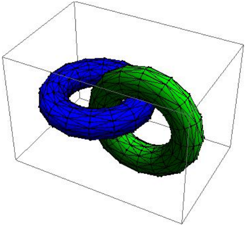

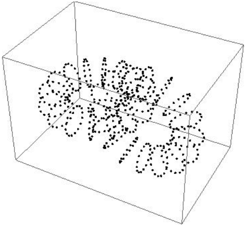

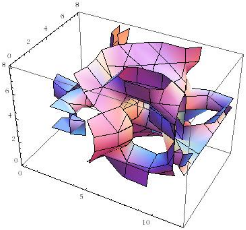

- The correctness can be checked by various topological invariants: number of connected components, dimension, number of “holes” (i.e. homology groups), etc.

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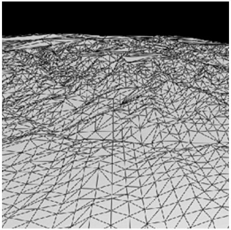

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- A *systematic analysis of the critical points* of f enables
  - a *topological reconstruction* of M as a CW complex with one cell of dimension i for each critical point of index i, where the index of a critical point is the number of independent directions in which the function values decrease, and

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  - a decomposition of the domain M into the *Morse-Smale complex* with respect to the flow lines of the gradient vector field of f

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- A critical cell of index i: a cell of dimension i, where there is no such exception.
- The discrete vector field on M induced by f: the set of all such exceptions, that is, the pairs (a, b) with $a \subset \partial b$ and $f(a) > f(b)$.

Advantages

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- It incorporates a very simple mechanism for dealing with noise in the data.

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- King, Knudsen, M. (2006): The definition of a parametric discrete Morse function and an algorithm for the bifurcation diagram of the critical cells

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- An obvious problem left to solve: simultaneous cancelling of pairs of critical cells.