Test for gradient fields.

Observe: if \( \vec{F} = M\hat{i} + N\hat{j} \) is a gradient field then \( N_x = M_y \). Indeed, if \( \vec{F} = \nabla f \) then \( M = f_x \), \( N = f_y \), so \( N_x = f_{yx} = f_{xy} = M_y \).

Claim: Conversely, if \( \vec{F} \) is defined and differentiable at every point of the plane, and \( N_x = M_y \), then \( \vec{F} = M\hat{i} + N\hat{j} \) is a gradient field.

Example: \( \vec{F} = -y\hat{i} + x\hat{j} \): \( N_x = 1, M_y = -1 \), so \( \vec{F} \) is not a gradient field.

Example: for which value(s) of \( a \) is \( \vec{F} = (4x^2 + axy)\hat{i} + (3y^2 + 4x^2)\hat{j} \) a gradient field? Answer: \( N_x = 8x, M_y = ax \), so \( a = 8 \).

Finding the potential: if above test says \( \vec{F} \) is a gradient field, we have 2 methods to find the potential function \( f \). Illustrated for the above example (taking \( a = 8 \)):

Method 1: using line integrals (FTC backwards):

We know that if \( C \) starts at \((0,0)\) and ends at \((x_1,y_1)\) then \( f(x_1,y_1) - f(0,0) = \int_C \vec{F} \cdot d\vec{r} \). Here \( f(0,0) \) is just an integration constant (if \( f \) is a potential then so is \( f + c \)). Can also choose the simplest \( C \) from \((0,0)\) to \((x_1,y_1)\).

Simplest choice: take \( C \) = portion of \( x \)-axis from \((0,0)\) to \((x_1,0)\), then vertical segment from \((x_1,0)\) to \((x_1,y_1)\) (picture drawn).

Then \( \int_C \vec{F} \cdot d\vec{r} = \int_{C_1+C_2} (4x^2 + 8xy) \, dx + (3y^2 + 4x^2) \, dy \):

Over \( C_1, 0 \leq x \leq x_1, y = 0 \), \( dy = 0 \): \( \int_{C_1} = \int_0^{x_1} (4x^2 + 8x \cdot 0) \, dx = \left[ \frac{4}{3} x^3 \right]_0^{x_1} = \frac{4}{3} x_1^3 \).

Over \( C_2, 0 \leq y \leq y_1, x = x_1 \), \( dx = 0 \): \( \int_{C_2} = \int_0^{y_1} (3y^2 + 4x_1^2) \, dy = \left[ y^3 + 4x_1^2 y \right]_0^{y_1} = y_1^3 + 4x_1^2 y_1 \).

So \( f(x_1,y_1) = \frac{4}{3} x_1^3 + y_1^3 + 4x_1^2 y_1 \) (+constant).

Method 2: using antiderivatives:

We want \( f(x,y) \) such that (1) \( f_x = 4x^2 + 8xy \), (2) \( f_y = 3y^2 + 4x^2 \).

Taking antiderivative of (1) w.r.t. \( x \) (treating \( y \) as a constant), we get \( f(x,y) = \frac{4}{3} x^3 + 4x^2 y + \text{integration constant} \) (independent of \( x \)). The integration constant still depends on \( y \), call it \( g(y) \).

So \( f(x,y) = \frac{4}{3} x^3 + 4x^2 y + g(y) \). Take partial w.r.t. \( y \), to get \( f_y = 4x^2 + g'(y) \).

Comparing this with (2), we get \( g'(y) = 3y^2 \), so \( g(y) = y^3 + c \).

Plugging into above formula for \( f \), we finally get \( f(x,y) = \frac{4}{3} x^3 + 4x^2 y + y^3 + c \).

Curl.

Now we have: \( N_x = M_y \iff^{*} \vec{F} \) is a gradient field \( \iff \vec{F} \) is conservative: \( \oint_C \vec{F} \cdot d\vec{r} = 0 \) for any closed curve.

\( (*) \): \( \iff \) only holds if \( \vec{F} \) is defined everywhere, or in a “simply-connected” region – see next week.

Failure of conservativeness is given by the curl of \( \vec{F} \):

Definition: \( \text{curl}(\vec{F}) = N_x - M_y \).

Interpretation of curl: for a velocity field, curl = (twice) angular velocity of the rotation component of the motion.
(Ex: \( \vec{F} = (a, b) \) uniform translation, \( \vec{F} = (x, y) \) expanding motion have curl zero; whereas \( \vec{F} = (-y, x) \) rotation at unit angular velocity has curl = 2).

For a force field, \( \text{curl} \vec{F} = \text{torque exerted on a test mass, measures how} \vec{F} \text{ imparts rotation motion.} \)

For translation motion: \[
\frac{\text{Force}}{\text{Mass}} = \text{acceleration} = \frac{d}{dt}(\text{velocity}).
\]

For rotation effects: \[
\frac{\text{Torque}}{\text{Moment of inertia}} = \text{angular acceleration} = \frac{d}{dt}(\text{angular velocity}).
\]

18.02 Lecture 22. – Thu, Nov 1, 2007

Handouts: PS8 solutions, PS9, practice exams 3A and 3B.

Green’s theorem.

If \( C \) is a positively oriented closed curve enclosing a region \( R \), then
\[
\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl} \vec{F} \ dA \quad \text{which means} \quad \oint_C (M \, dx + N \, dy) = \iint_R (N_x - M_y) \, dA.
\]

Example (reduce a complicated line integral to an easy \( \iint \)):
Let \( C \) = unit circle centered at (2,0), counterclockwise. \( R \) = unit disk at (2,0). Then
\[
\oint_C ye^{-x} \, dx + \left( \frac{1}{2}x^2 - e^{-x} \right) \, dy = \iint_R N_x - M_y \, dA = \iint_R (x + e^{-x}) - e^{-x} \, dA = \iint_R x \, dA.
\]
This is equal to area \( \cdot \bar{x} = \pi \cdot 2 = 2\pi \) (or by direct computation of the iterated integral). (Note: direct calculation of the line integral would probably involve setting \( x = 2 + \cos \theta \), \( y = \sin \theta \), but then calculations get really complicated.)

Application: proof of our criterion for gradient fields.

Theorem: if \( \vec{F} = Mi + Nj \) is defined and continuously differentiable in the whole plane, then \( N_x = M_y \Rightarrow \vec{F} \) is conservative (\( \Leftrightarrow \vec{F} \) is a gradient field).

If \( N_x = M_y \), then by Green, \( \oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl} \vec{F} \, dA = \iint_R 0 \, dA = 0 \). So \( \vec{F} \) is conservative.

Note: this only works if \( \vec{F} \) and its curl are defined everywhere inside \( R \). For the vector field on PS8 Problem 2, we can’t do this if the region contains the origin – for example, the line integral along the unit circle is non-zero even though \( \text{curl(\vec{F})} \) is zero wherever it’s defined.

Proof of Green’s theorem. 2 preliminary remarks:
1) the theorem splits into two identities, \( \oint_C M \, dx = -\iint_R M_y \, dA \) and \( \oint_C N \, dy = \iint_R N_x \, dA \).
2) additivity: if the theorem is true for \( R_1 \) and \( R_2 \) then it’s true for the union \( R = R_1 \cup R_2 \) (picture shown): \( \oint_C = \oint_{C_1} + \oint_{C_2} \) (the line integrals along inner portions cancel out) and \( \iint_R = \iint_{R_1} + \iint_{R_2} \).

Main step in the proof: prove \( \oint_C M \, dx = -\iint_R M_y \, dA \) for “vertically simple” regions: \( a < x < b, f_0(x) < y < f_1(x) \). (picture drawn). This involves calculations similar to PS5 Problem 3.

LHS: break \( C \) into four sides (\( C_1 \) lower, \( C_2 \) right vertical segment, \( C_3 \) upper, \( C_4 \) left vertical segment); \( \oint_{C_2} M \, dx = \oint_{C_4} M \, dx = 0 \) since \( x = \) constant on \( C_2 \) and \( C_4 \). So
\[
\oint_C = \oint_{C_1} + \oint_{C_3} = \int_a^b M(x, f_0(x)) \, dx - \int_a^b M(x, f_1(x)) \, dx
\]
(usually along \( C_1 \): parameter \( a \leq x \leq b, y = f_0(x) \); along \( C_2 \), \( x \) from \( b \) to \( a \), hence - sign; \( y = f_1(x) \)).
\[
\text{RHS: } - \iiint_R M_y \, dA = - \int_a^b \int_{f_0(x)}^{f_1(x)} M_y \, dy \, dx = - \int_a^b (M(x, f_1(x)) - M(x, f_0(x))) \, dx = \text{LHS}.
\]

Finally observe: any region \( R \) can be subdivided into vertically simple pieces (picture shown); for each piece \( \mathcal{C}_i \), \( M \, dx = - \iiint_{\mathcal{C}_i} M_y \, dA \), so by additivity \( \mathcal{C}_i \, M \, dx = - \iiint_R M_y \, dA \).

Similarly \( \mathcal{C}_i \, N \, dy = \iiint_R \! N_x \, dA \) by subdividing into horizontally simple pieces. This completes the proof.

**Example.** The area of a region \( R \) can be evaluated using a line integral: for example, \( \oint_C x \, dy = \int_R \! 1 \, dA = \text{area}(R) \).

This idea was used to build mechanical devices that measure area of arbitrary regions on a piece of paper: planimeters (photo of the actual object shown, and principle explained briefly: as one moves its arm along a closed curve, the planimeter calculates the line integral of a suitable vector field by means of an ingenious mechanism; at the end of the motion, one reads the area).

18.02 Lecture 23. – Fri, Nov 2, 2007

**Flux.** The flux of a vector field \( \vec{F} \) across a plane curve \( C \) is \( \int_C \vec{F} \cdot \hat{n} \, ds \), where \( \hat{n} \) is normal vector to \( C \), rotated \( 90^\circ \) clockwise from \( \hat{T} \).

We now have two types of line integrals: work, \( \int \vec{F} \cdot \hat{T} \, ds \), sums \( \vec{F} \cdot \hat{T} \) = component of \( \vec{F} \) in direction of \( C \), along the curve \( C \). Flux, \( \int \vec{F} \cdot \hat{n} \, ds \), sums \( \vec{F} \cdot \hat{n} \) = component of \( \vec{F} \) perpendicular to \( C \), along the curve.

If we break \( C \) into small pieces of length \( \Delta s \), the flux is \( \sum_i (\vec{F} \cdot \hat{n}) \Delta s_i \).

**Physical interpretation:** if \( \vec{F} \) is a velocity field (e.g. flow of a fluid), flux measures how much matter passes through \( C \) per unit time.

Look at a small portion of \( C \): locally \( \vec{F} \) is constant, what passes through portion of \( C \) in unit time is contents of a parallelogram with sides \( \Delta s \) and \( \vec{F} \) (picture shown with \( \vec{F} \) horizontal, and portion of curve = diagonal line segment). The area of this parallelogram is \( \Delta s \cdot \text{height} = \Delta s \, (\vec{F} \cdot \hat{n}). \) (picture shown rotated with portion of \( C \) horizontal, at base of parallelogram). Summing these contributions along all of \( C \), we get that \( \int (\vec{F} \cdot \hat{n}) \, ds \) is the total flow through \( C \) per unit time; counting positively what flows towards the right of \( C \), negatively what flows towards the left of \( C \), as seen from the point of view of a point travelling along \( C \).

**Example:** \( C = \) circle of radius \( a \) counterclockwise, \( \vec{F} = xi + yj \) (picture shown): along \( C \), \( \vec{F} /\hat{n} \), and \( |\vec{F}| = a \), so \( \vec{F} \cdot \hat{n} = a \). So

\[
\int_C \vec{F} \cdot \hat{n} \, ds = \int_C a \, ds = a \text{length}(C) = 2\pi a^2.
\]

Meanwhile, the flux of \(-yi + xj\) across \( C \) is zero (field tangent to \( C \)).

That was a geometric argument. What about the general situation when calculation of the line integral is required?

Observe: \( d\vec{r} = \hat{T} \, ds = (dx, dy) \), and \( \hat{n} \) is \( \hat{T} \) rotated \( 90^\circ \) clockwise; so \( \hat{n} \, ds = (dy, -dx) \).

So, if \( \vec{F} = Pi + Qj \) (using new letters to make things look different; of course we could call the components \( M \) and \( N \)), then

\[
\int_C \vec{F} \cdot \hat{n} \, ds = \int_C (P, Q) \cdot (dy, -dx) = \int_C -Q \, dx + P \, dy.
\]
(or if \( \vec{F} = \langle M, N \rangle, \int_C -N \, dx + M \, dy \)).

So we can compute flux using the usual method, by expressing \( x, y, dx, dy \) in terms of a parameter variable and substituting (no example given).

**Green’s theorem for flux.** If \( C \) encloses \( R \) counterclockwise, and \( \vec{F} = P\hat{i} + Q\hat{j} \), then

\[
\oint_C \vec{F} \cdot \hat{n} \, ds = \iint_R \text{div}(\vec{F}) \, dA, \quad \text{where} \quad \text{div}(\vec{F}) = P_x + Q_y \quad \text{is the divergence of \( \vec{F} \).}
\]

Note: the counterclockwise orientation of \( C \) means that we count flux of \( \vec{F} \) out of \( R \) through \( C \).

Proof: \( \oint_C \vec{F} \cdot \hat{n} \, ds = \oint_C -Q \, dx + P \, dy \). Call \( M = -Q \) and \( N = P \), then apply usual Green’s theorem \( \oint_C M \, dx + N \, dy = \iint_R (N_x - M_y) \, dA \) to get

\[
\oint_C -Q \, dx + P \, dy = \iint_R (P_x - (-Q_y)) \, dA = \iint_R \text{div}(\vec{F}) \, dA.
\]

This proof by “renaming” the components is why we called the components \( P, Q \) instead of \( M, N \). If we call \( \vec{F} = \langle M, N \rangle \) the statement becomes \( \oint_C -N \, dx + M \, dy = \iint_R (M_x + N_y) \, dA \).

**Example:** in the above example \( (x\hat{i} + y\hat{j} \) across circle), \( \text{div} \vec{F} = 2 \), so flux \( = \iint_R 2 \, dA = 2 \text{area}(R) = 2\pi a^2 \). If we translate \( C \) to a different position (not centered at origin) (picture shown) then direct calculation of flux is harder, but total flux is still \( 2\pi a^2 \).

Physical interpretation: in an incompressible fluid flow, divergence measures source/sink density/rate, i.e. how much fluid is being added to the system per unit area and per unit time.