18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 21. – **Tue, Oct 30, 2007**

Test for gradient fields.

Observe: if $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ is a gradient field then $N_x = M_y$. Indeed, if $\vec{F} = \nabla f$ then $M = f_x$, $N = f_y$, so $N_x = f_{yx} = f_{xy} = M_y$.

Claim: Conversely, if $\vec{F}$ is defined and differentiable at every point of the plane, and $N_x = M_y$, then $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ is a gradient field.

Example: $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$: $N_x = 1$, $M_y = -1$, so $\vec{F}$ is not a gradient field.

Example: for which value(s) of a is $\vec{F} = (4x^2 + axy)\hat{\imath} + (3y^2 + 4x^2)\hat{\jmath}$ a gradient field? Answer: $N_x = 8x$, $M_y = ax$, so $a = 8$.

Finding the potential: if above test says $\vec{F}$ is a gradient field, we have 2 methods to find the potential function f. Illustrated for the above example (taking $a = 8$):

Method 1: using line integrals (FTC backwards):

We know that if C starts at $(0,0)$ and ends at (x_1, y_1) then $f(x_1, y_1) - f(0,0) = \int_C \vec{F} \cdot d\vec{r}$. Here $f(0,0)$ is just an integration constant (if f is a potential then so is $f + c$). Can also choose the simplest curve C from $(0,0)$ to (x_1, y_1).

Simplest choice: take C = portion of x-axis from $(0,0)$ to $(x_1, 0)$, then vertical segment from $(x_1, 0)$ to (x_1, y_1) (picture drawn).

Then $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_{C_1 + C_2} (4x^2 + 8xy)\,dx + (3y^2 + 4x^2)\,dy$:

Over C_1, $0 \le x \le x_1$, $y = 0$, $dy = 0$: $\displaystyle\int_{C_1} = \int_0^{x_1} (4x^2 + 8x \cdot 0)\,dx = \left[\frac{4}{3}x^3\right]_0^{x_1} = \frac{4}{3}x_1^3.$

Over C_2, $0 \le y \le y_1$, $x = x_1$, $dx = 0$: $\displaystyle\int_{C_2} = \int_0^{y_1} (3y^2 + 4x_1^2)\,dy = \left[y^3 + 4x_1^2 y\right]_0^{y_1} = y_1^3 + 4x_1^2 y_1.$

So $f(x_1, y_1) = \dfrac{4}{3}x_1^3 + y_1^3 + 4x_1^2 y_1$ (+constant).

Method 2: using antiderivatives:

We want $f(x, y)$ such that (1) $f_x = 4x^2 + 8xy$, (2) $f_y = 3y^2 + 4x^2$.

Taking antiderivative of (1) w.r.t. x (treating y as a constant), we get $f(x, y) = \frac{4}{3}x^3 + 4x^2 y +$ integration constant (independent of x). The integration constant still depends on y, call it $g(y)$.

So $f(x, y) = \frac{4}{3}x^3 + 4x^2 y + g(y)$. Take partial w.r.t. y, to get $f_y = 4x^2 + g'(y)$.

Comparing this with (2), we get $g'(y) = 3y^2$, so $g(y) = y^3 + c$.

Plugging into above formula for f, we finally get $f(x, y) = \frac{4}{3}x^3 + 4x^2 y + y^3 + c$.

Curl.

Now we have: $N_x = M_y \Leftrightarrow^* \vec{F}$ is a gradient field $\Leftrightarrow \vec{F}$ is conservative: $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve.

(*): $\Rightarrow$ only holds if $\vec{F}$ is defined everywhere, or in a "simply-connected" region – see next week.

Failure of conservativeness is given by the *curl* of $\vec{F}$:

Definition: $\text{curl}(\vec{F}) = N_x - M_y$.

Interpretation of curl: for a velocity field, curl = (twice) angular velocity of the rotation component of the motion.

(Ex: $\vec{F} = \langle a, b\rangle$ uniform translation, $\vec{F} = \langle x, y\rangle$ expanding motion have curl zero; whereas $\vec{F} = \langle -y, x\rangle$ rotation at unit angular velocity has curl $= 2$).

For a force field, curl $\vec{F} =$ torque exerted on a test mass, measures how $\vec{F}$ imparts rotation motion.

For translation motion: $\dfrac{\text{Force}}{\text{Mass}} = \text{acceleration} = \dfrac{d}{dt}(\text{velocity})$.

For rotation effects: $\dfrac{\text{Torque}}{\text{Moment of inertia}} = \text{angular acceleration} = \dfrac{d}{dt}(\text{angular velocity})$.

18.02 Lecture 22. – **Thu, Nov 1, 2007**

Handouts: PS8 solutions, PS9, practice exams 3A and 3B.

Green's theorem.

If C is a positively oriented closed curve enclosing a region R, then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\, dA \quad \text{which means} \quad \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA.$$

Example (reduce a complicated line integral to an easy $\iint$):
Let $C =$ unit circle centered at (2,0), counterclockwise. $R =$ unit disk at $(2,0)$. Then

$$\oint_C ye^{-x}\,dx + (\frac{1}{2}x^2 - e^{-x})\,dy = \iint_R N_x - M_y\,dA = \iint_R (x + e^{-x}) - e^{-x}\,dA = \iint_R x\,dA.$$

This is equal to area $\cdot\, \bar{x} = \pi \cdot 2 = 2\pi$ (or by direct computation of the iterated integral). (Note: direct calculation of the line integral would probably involve setting $x = 2 + \cos\theta$, $y = \sin\theta$, but then calculations get really complicated.)

Application: proof of our criterion for gradient fields.

Theorem: if $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ is defined and continuously differentiable in the whole plane, then $N_x = M_y \Rightarrow \vec{F}$ is conservative ($\Leftrightarrow \vec{F}$ is a gradient field).

If $N_x = M_y$ then by Green, $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\,dA = \iint_R 0\,dA = 0$. So $\vec{F}$ is conservative.

Note: this only works if $\vec{F}$ and its curl are defined everywhere inside R. For the vector field on PS8 Problem 2, we can't do this if the region contains the origin – for example, the line integral along the unit circle is non-zero even though curl$(\vec{F})$ is zero wherever it's defined.

Proof of Green's theorem. 2 preliminary remarks:

1) the theorem splits into two identities, $\oint_C M\,dx = -\iint_R M_y\,dA$ and $\oint_C N\,dy = \iint_R N_x\,dA$.

2) additivity: if theorem is true for R_1 and R_2 then it's true for the union $R = R_1 \cup R_2$ (picture shown): $\oint_C = \oint_{C_1} + \oint_{C_2}$ (the line integrals along inner portions cancel out) and $\iint_R = \iint_{R_1} + \iint_{R_2}$.

Main step in the proof: prove $\oint_C M\,dx = -\iint_R M_y\,dA$ for "vertically simple" regions: $a < x < b$, $f_0(x) < y < f_1(x)$. (picture drawn). This involves calculations similar to PS5 Problem 3.

LHS: break C into four sides (C_1 lower, C_2 right vertical segment, C_3 upper, C_4 left vertical segment); $\int_{C_2} M\,dx = \int_{C_4} M\,dx = 0$ since $x =$constant on C_2 and C_4. So

$$\oint_C = \int_{C_1} + \int_{C_3} = \int_a^b M(x, f_0(x))\,dx - \int_a^b M(x, f_1(x))\,dx$$

(using along C_1: parameter $a \le x \le b$, $y = f_0(x)$; along C_2, x from b to a, hence $-$ sign; $y = f_1(x)$).

RHS: $-\iint_R M_y\,dA = -\int_a^b \int_{f_0(x)}^{f_1(x)} M_y\,dy\,dx = -\int_a^b (M(x, f_1(x)) - M(x, f_0(x))\,dx$ (= LHS).

Finally observe: any region R can be subdivided into vertically simple pieces (picture shown); for each piece $\oint_{C_i} M\,dx = -\iint_{R_i} M_y\,dA$, so by additivity $\oint_C M\,dx = -\iint_R M_y\,dA$.

Similarly $\oint_C N\,dy = \iint_R N_x\,dA$ by subdividing into horizontally simple pieces. This completes the proof.

Example. The area of a region R can be evaluated using a line integral: for example, $\oint_C x\,dy = \iint_R 1 dA = area(R)$.

This idea was used to build mechanical devices that measure area of arbitrary regions on a piece of paper: planimeters (photo of the actual object shown, and principle explained briefly: as one moves its arm along a closed curve, the planimeter calculates the line integral of a suitable vector field by means of an ingenious mechanism; at the end of the motion, one reads the area).

18.02 Lecture 23. – **Fri, Nov 2, 2007**

Flux. The flux of a vector field $\vec{F}$ across a plane curve C is $\int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds$, where $\hat{\boldsymbol{n}}$ = normal vector to C, rotated 90° clockwise from $\hat{\boldsymbol{T}}$.

We now have two types of line integrals: work, $\int \vec{F}\cdot\hat{\boldsymbol{T}}\,ds$, sums $\vec{F}\cdot\hat{\boldsymbol{T}}$ = component of $\vec{F}$ in direction of C, along the curve C. Flux, $\int \vec{F}\cdot\hat{\boldsymbol{n}}\,ds$, sums $\vec{F}\cdot\hat{\boldsymbol{n}}$ = component of $\vec{F}$ perpendicular to C, along the curve.

If we break C into small pieces of length Δs, the flux is $\sum_i (\vec{F}\cdot\hat{\boldsymbol{n}})\,\Delta s_i$.

Physical interpretation: if $\vec{F}$ is a velocity field (e.g. flow of a fluid), flux measures how much matter passes through C per unit time.

Look at a small portion of C: locally $\vec{F}$ is constant, what passes through portion of C in unit time is contents of a parallelogram with sides Δs and $\vec{F}$ (picture shown with $\vec{F}$ horizontal, and portion of curve = diagonal line segment). The area of this parallelogram is $\Delta s\cdot$height $= \Delta s\,(\vec{F}\cdot\hat{\boldsymbol{n}})$. (picture shown rotated with portion of C horizontal, at base of parallelogram). Summing these contributions along all of C, we get that $\int (\vec{F}\cdot\hat{\boldsymbol{n}})\,ds$ is the total flow through C per unit time; counting positively what flows towards the right of C, negatively what flows towards the left of C, as seen from the point of view of a point travelling along C.

Example: C = circle of radius a counterclockwise, $\vec{F} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ (picture shown): along C, $\vec{F}//\hat{\boldsymbol{n}}$, and $|\vec{F}| = a$, so $\vec{F}\cdot\hat{\boldsymbol{n}} = a$. So

$$\int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \int_C a\,ds = a\,\text{length}(C) = 2\pi a^2.$$

Meanwhile, the flux of $-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$ across C is zero (field tangent to C).

That was a geometric argument. What about the general situation when calculation of the line integral is required?

Observe: $d\vec{r} = \hat{\boldsymbol{T}}\,ds = \langle dx, dy\rangle$, and $\hat{\boldsymbol{n}}$ is $\hat{\boldsymbol{T}}$ rotated 90° clockwise; so $\hat{\boldsymbol{n}}\,ds = \langle dy, -dx\rangle$.

So, if $\vec{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}$ (using new letters to make things look different; of course we could call the components M and N), then

$$\int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \int_C \langle P, Q\rangle\cdot\langle dy, -dx\rangle = \int_C -Q\,dx + P\,dy.$$

(or if $\vec{F} = \langle M, N \rangle$, $\int_C -N\,dx + M\,dy$).

So we can compute flux using the usual method, by expressing x, y, dx, dy in terms of a parameter variable and substituting (no example given).

Green's theorem for flux. If C encloses R counterclockwise, and $\vec{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}$, then

$$\oint_C \vec{F} \cdot \hat{\boldsymbol{n}}\,ds = \iint_R \operatorname{div}(\vec{F})\,dA, \quad \text{where} \quad \operatorname{div}(\vec{F}) = P_x + Q_y \quad \text{is the divergence of } \vec{F}.$$

Note: the counterclockwise orientation of C means that we count flux of $\vec{F}$ *out* of R through C.

Proof: $\displaystyle\oint_C \vec{F} \cdot \hat{\boldsymbol{n}}\,ds = \oint_C -Q\,dx + P\,dy$. Call $M = -Q$ and $N = P$, then apply usual Green's theorem $\displaystyle\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$ to get

$$\oint_C -Q\,dx + P\,dy = \iint_R (P_x - (-Q_y))\,dA = \iint_R \operatorname{div}(\vec{F})\,dA.$$

This proof by "renaming" the components is why we called the components P, Q instead of M, N. If we call $\vec{F} = \langle M, N \rangle$ the statement becomes $\displaystyle\oint_C -N\,dx + M\,dy = \iint_R (M_x + N_y)\,dA.$

Example: in the above example ($x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ across circle), $\operatorname{div}\vec{F} = 2$, so flux $= \iint_R 2\,dA = 2\,\operatorname{area}(R) = 2\pi a^2$. If we translate C to a different position (not centered at origin) (picture shown) then direct calculation of flux is harder, but total flux is still $2\pi a^2$.

Physical interpretation: in an incompressible fluid flow, divergence measures source/sink density/rate, i.e. how much fluid is being added to the system per unit area and per unit time.