18.02 Multivariable Calculus
Fall 2007
18.02 Lecture 16. – Thu, Oct 18, 2007

Handouts: PS6 solutions, PS7.

Double integrals.
Recall integral in 1-variable calculus: \( \int_a^b f(x) \, dx = \) area below graph \( y = f(x) \) over \([a, b]\).
Now: double integral \( \iint_R f(x, y) \, dA = \) volume below graph \( z = f(x, y) \) over plane region \( R \).
Cut \( R \) into small pieces \( \Delta A \Rightarrow \) the volume is approximately \( \sum f(x_i, y_i) \Delta A_i \). Limit as \( \Delta A \to 0 \) gives \( \iint_R f(x, y) \, dA \). (picture shown)

How to compute the integral? By taking slices: \( S(x) = \) area of the slice by a plane parallel to \( yz \)-plane (picture shown): then

\[
\text{volume} = \int_{x_{\text{min}}}^{x_{\text{max}}} S(x) \, dx,
\]
and for given \( x \), \( S(x) = \int f(x, y) \, dy \).

In the inner integral, \( x \) is a fixed parameter, \( y \) is the integration variable. We get an \textit{iterated integral}.

Example 1: \( z = 1 - x^2 - y^2 \), region \( 0 \leq x \leq 1, \ 0 \leq y \leq 1 \) (picture shown):

\[
\int_0^1 \int_0^1 (1 - x^2 - y^2) \, dy \, dx.
\]

(note: \( dA = dy \, dx \), limit of \( \Delta A = \Delta y \Delta x \) for small rectangles).

How to evaluate:
1) inner integral (\( x \) is constant): \( \int_0^1 (1 - x^2 - y^2) \, dy = \left[ (1 - x^2)y - \frac{1}{3}y^3 \right]_0^1 = (1 - x^2) - \frac{1}{3} = \frac{2}{3} - x^2 \).
2) outer integral: \( \int_0^1 \left( \frac{2}{3} - x^2 \right) \, dx = \left[ \frac{2}{3}x - \frac{1}{3}x^3 \right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \).

Example 2: same function over the quarter disc \( R: x^2 + y^2 < 1 \) in the first quadrant.

How to find the bounds of integration? Fix \( x \) constant: what is a slice parallel to \( y \)-axis? bounds for \( y = \) from \( y = 0 \) to \( y = \sqrt{1 - x^2} \) in the inner integral. For the outer integral: first slice is \( x = 0 \), last slice is \( x = 1 \). So we get:

\[
\int_0^1 \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2) \, dy \, dx.
\]

(note the inner bounds depend on the outer variable \( x \); the outer bounds are constants!)

Inner: \( \left[ (1 - x^2)y - y^3/3 \right]_0^{\sqrt{1-x^2}} = \frac{2}{3}(1-x^2)^{3/2} \).

Outer: \( \int_0^1 \frac{2}{3}(1-x^2)^{3/2} \, dx = \cdots = \frac{\pi}{8} \).

\( \cdots \) = trig. substitution \( x = \sin \theta, \ dx = \cos \theta \, d\theta, \ (1-x^2)^{3/2} = \cos^3 \theta \). Then use double angle formulas... complicated! I carried out part of the calculation to show how it would be done but then stopped before the end to save time; students may be confused about what happened exactly.)

Exchanging order of integration.
\( \int_0^1 \int_0^2 dy \, dx = \int_0^2 \int_0^1 dy \, dx \), since region is a rectangle (shown). In general, more complicated!
Example 3: $\int_0^1 \int_x^{\sqrt{x}} \frac{e^y}{y} \, dy \, dx$: inner integral has no formula. To exchange order:

1) draw the region (here: $x < y < \sqrt{x}$ for $0 \leq x \leq 1$ – picture drawn on blackboard).
2) figure out bounds in other direction: fixing a value of $y$, what are the bounds for $x$? here: left border is $x = y^2$, right is $x = y$; first slice is $y = 0$, last slice is $y = 1$, so we get

$$\int_0^1 \int_0^{y^2} \frac{e^y}{y} \, dx \, dy = \int_0^1 \frac{e^y}{y} (y - y^2) \, dy = \int_0^1 e^y - ye^y \, dy = [-ye^y + 2e^y]_0^1 = e - 2.$$ 

(the last integration can be done either by parts, or by starting from the guess $-ye^y$ and adjusting.)

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Integration in polar coordinates. ($x = r \cos \theta, \ y = r \sin \theta$): useful if either integrand or region have a simpler expression in polar coordinates.

Area element: $\Delta A \simeq (r \Delta \theta) \Delta r$ (picture drawn of a small element with sides $\Delta r$ and $r \Delta \theta$). Taking $\Delta \theta, \Delta r \to 0$, we get $dA = r \, dr \, d\theta$.

Example (same as last time): $\int \int_{x^2 + y^2 \leq 1, \ x \geq 0, \ y \geq 0} (1 - x^2 - y^2) \, dx \, dy = \int_0^{\pi/2} \int_0^1 (1 - r^2) \, r \, dr \, d\theta$.

Inner: $\left[ \frac{1}{2} r^2 - \frac{1}{4} r^4 \right]_0^1 = \frac{1}{4}$. Outer: $\frac{\pi}{2} \frac{1}{2} = \frac{\pi}{4}$.

In general: when setting up $\int \int f(r \, dr \, d\theta)$, find bounds as usual: given a fixed $\theta$, find initial and final values of $r$ (sweep region by rays).

Applications.

1) The area of the region $R$ is $\iint_R 1 \, dA$. Also, the total mass of a planar object with density $\delta = \lim_{\Delta A \to 0} \Delta m/\Delta A$ (mass per unit area, $\delta = \delta(x,y)$ – if uniform material, constant) is given by:

$$M = \iiint_R \delta \, dA.$$ 

2) recall the average value of $f$ over $R$ is $\bar{f} = \frac{1}{\text{Area}} \iint_R f \, dA$. The center of mass, or centroid, of a plate with density $\delta$ is given by weighted average

$$\bar{x} = \frac{1}{\text{mass}} \iint_R x \delta \, dA, \quad \bar{y} = \frac{1}{\text{mass}} \iint_R y \delta \, dA$$

3) moment of inertia: physical equivalent of mass for rotational motion. (mass = how hard it is to impart translation motion; moment of inertia about some axis = same for rotation motion around that axis)

Idea: kinetic energy for a single mass $m$ at distance $r$ rotating at angular speed $\omega = d\theta/dt$ (so velocity $v = r\omega$) is $\frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$, $I_0 = mr^2$ is the moment of inertia.

For a solid with density $\delta$, $I_0 = \iiint_R r^2 \delta \, dA$ (moment of inertia / origin). (the rotational energy is $\frac{1}{2}I_0\omega^2$).
Moment of inertia about an axis: \( I = \iint_R (\text{distance to axis})^2 \delta \, dA \). E.g. about \( x \)-axis, distance is \(|y|\), so

\[
I_x = \iint_R y^2 \delta \, dA.
\]

Examples: 1) disk of radius \( a \) around its center (\( \delta = 1 \)):

\[
I_0 = \int_0^{2\pi} \int_0^a r^2 r \, dr \, d\theta = 2\pi \left[ \frac{r^4}{4} \right]_0^a = \frac{\pi a^4}{2}.
\]

2) same disk, about a point on the circumference?

Setup: place origin at point so integrand is easier; diameter along \( x \)-axis; then polar equation of circle is \( r = 2a \cos \theta \) (explained on a picture). Thus

\[
I_0 = \int_{-\pi/2}^{\pi/2} \int_0^{2a \cos \theta} r^2 r \, dr \, d\theta = \cdots = \frac{3}{2} \pi a^4.
\]