3D visibility and Lines in space

Sylvain Lazard

INRIA Nancy – Vegas Project-team
3D visibility computations are central in many applications (computer graphics, robotics, manufacturing, etc.)

Well-solved in practice using approximation and hardware

Still, 3D visibility computations are extremely costly

Two typical difficult problems:
- Surface to surface visibility queries
- Shadows - limits of umbra and penumbra

for which there is no satisfactory solution
Simulating illumination is easier if the shadows are known.

Lischinski et al. [Siggraph 93]
Shadow boundaries

2D

light

Obstacles
Shadow boundaries

2D

light

full light
Shadow boundaries

2D

light

full light

umbra

penumbra

umbra

penumbra

unbra

penumbra

penumbra

umbra
Shadow boundaries

3D

- light source
- penumbra
- full light
- umbra
- penumbra

EV

EEE
Shadow boundaries

3D

light source

penumbra

full light

EV

VEE

umbra

penumbra

EEE

EEEEE
Shadow boundaries

3D

light source

penumbra

full light

EV

VEE, VV

umbra

penumbra

EEE

EEEE, FEE
Shadow boundaries

3D

Visibility  EV  VEE, VV
Skeleton  EEE  EEEE, FEE

[DDP 97]
Shadow boundaries

3D

light source

penumbra

full light

umbra

penumbra

Visibility
Skeleton

EV
VEE, VV

EEE
EEEE, FEE

in line space

(Embedding of the vis. skeleton in $\mathbb{R}^3 \sim$ Aspect graph)
Visibility skeleton: past work

Key space: space of maximal free line segments
Visibility skeleton: past work

**Key space:** space of maximal free line segments

**Visibility complex:**
- 2D: [Pocchiola, Vegter 96]
- 3D: [Durand, Drettakis, Puech 02]

Partition of this space into connected components of segments tangent to the same objects
Visibility skeleton: past work

Key space: space of maximal free line segments

Visibility complex: 2D: [Pocchiola, Vegter 96] 3D: [Durand, Drettakis, Puech 02]

Partition of this space into connected components of segments tangent to the same objects

Visibility skeleton: [Durand, Drettakis, Puech 97, 99]

1-skeleton of the complex

Arc: connected comp. of max. free segments tangent to 3 objects

Vertex: max. free line segment tangent to 4 objects
Visibility skeleton: past work

Proof-of-concept implementation [DDP 97,99]

Improved quality and comput. time

Algorithmics issues

Enumeration $\Theta(n^5)$

($+$ heuristics $\rightsquigarrow$ observed $\Theta(n^{2.5})$)

Robustness issues

$\rightsquigarrow n < 1500$
Visibility skeleton: past work

Proof-of-concept implementation [DDP 97,99]

Improved quality and comput. time

Algorithmics issues

Enumeration $\Theta(n^5)$

(+ heuristics $\leadsto$ observed $\Theta(n^{2.5})$)

Robustness issues

$\leadsto n < 1500$

2D section of the skeleton [DD 02]

Point light source,

possibly at $\infty$

$n \leadsto 100\ 000$
Problem:

Compute efficiently and robustly the visibility skeleton of polyhedra
Problem:

Compute efficiently and robustly the visibility skeleton of polyhedra

What is the size of this structure?
A number of problems with lines in 3D

Number of lines or max. free segments tangent to

\[ 4 \in k \text{ polytopes?} \]
A number of problems with lines in 3D

Number of lines or max. free segments tangent to

$4 \in \kappa$ polytopes?

4 polytopes?
A number of problems with lines in 3D

Number of lines or max. free segments tangent to

\[ 4 \in k \text{ polytopes?} \]

4 polytopes?

4 triangles?
A number of problems with lines in 3D

Number of lines or max. free segments tangent to

$4 \in k$ polytopes?

4 polytopes?

4 triangles?

4 segments?
A number of problems with lines in 3D

Number of lines or max. free segments tangent to

- $4 \in k$ polytopes?
- 4 polytopes?
- 4 triangles?
- 4 segments?
- 4 lines? Well-known
A number of problems with lines in 3D

Number of lines or max. free segments tangent to

4 \in k \text{ polytopes?}

4 polytopes?

4 triangles?

4 segments?

4 lines?

4 \in k \text{ balls?}
A number of problems with lines in 3D

Number of lines or max. free segments tangent to

\[ 4 \in \kappa \] polytopes?

4 polytopes?

4 triangles?

4 segments?

4 lines?

4 \in \kappa \] balls?

4 balls?
A number of problems with lines in 3D

Number of lines or max. free segments tangent to

4 ∈ k polytopes?
4 polytopes?
4 triangles?
4 segments?
4 lines?
4 ∈ k balls?
4 balls?

Efficient algorithms?
A number of problems with lines in 3D

Number of lines or max. free segments tangent to

\[ 4 \in k \] polytopes?

4 polytopes?

4 triangles?

4 segments?

4 lines?

4 \[ \in k \] balls?

4 balls?

Worst-case? Expected? Observed?

Efficient algorithms?
A number of problems with lines in 3D

Number of lines or max. free segments tangent to

\[4 \in_k \text{ polytopes?}\]

\[4 \text{ polytopes?}\]

\[4 \text{ triangles?}\]

\[4 \text{ segments?}\]

\[4 \text{ lines?}\]

\[4 \in_k \text{ balls?}\]

\[4 \text{ balls?}\]

Worst-case? Expected? Observed?

Efficient algorithms?

Robustness: degeneracies and predicates?
A number of problems with lines in 3D

Number of lines or max. free segments tangent to

\[4 \in k\] polytopes? \[\text{Bronnimann et al. 07}, \text{Efrat et al. 07}\]

4 polytopes? \[\text{Bronnimann et al. 07}, \text{Efrat et al. 07}\]

4 triangles? \[\text{Bronnimann et al. 07}\]

4 segments? \[\text{Bronnimann et al. 05}\]

4 lines? \textbf{Well-known}

4 \(\in k\) balls? \[\text{Devillers et al. 03}, \text{Glisse 07}\]

4 balls? \[\text{Macdonald et al. 01}, \ldots, \text{Borcea et al. 06}\]

Efficient algorithms?

\[\text{Durand et al. 02}, \text{Bronnimann et al. 07}, \text{Efrat et al. 07}, \text{Zhang et al. 08}\]

Robustness: degeneracies and predicates?

\[\text{Borcea et al. 06}, \text{Everett et al. 07}, \text{Devillers et al. 08}\]
Basic facts: **Line transversals to 3 lines**

3 pairwise skew lines lie on one ruling of a hyperbolic paraboloid or a hyperboloid of one sheet. Their transversals are the lines of the other ruling.
Basic facts: Line transversals to 4 lines

4 lines admit 0, 1, 2 or $\infty$ many transversals

The 4th line $L$ intersects $Q$ in

0, 1, 2 points $\sim\rightarrow$ 0, 1, 2 transversals

line $L \sim\rightarrow 1 \ (L)$ or $\infty$ many transversals (one ruling of $Q$)
Line transversals to 4 segments
Lines tangent to 4 objects

Line transversals to 4 segments

Intuition: at most 2 or $\infty$ ?
Lines tangent to 4 objects

Line transversals to 4 segments

Intuition: at most 2 or $\infty$? NO:

n segments admit up to n c.c. of transversals

(c.c. : when infinitely many a connected component in line space)

Line transversals to 4 segments

Intuition: at most 2 or $\infty$? NO:

$n$ segments admit up to $n$ c.c. of transversals

(c.c.: when infinitely many a connected component in line space)

n segments admit up to n c.c. of transversals

Proof sketch. Complete characterization of line transversals

Case: All segments in one ruling of a hyp. of one sheet
Lines in the other ruling: parameterized by points on a circle
Transversals to each segment: parameterized by an interval
Intersection of $n$ intervals on a circle: union of at most $n$ intervals

Lines tangent to 4 objects

Lines tangent to 4 triangles

at most 2? 10? 300?
Lines tangent to 4 objects

Lines tangent to 4 triangles

4 random triangles with 40 tangents
(4 cases out of 5 millions trials)

Lines tangent to 4 objects

Lower bound: 62
Idea: Transform each blue line into a very thin triangle

4 random triangles with 40 tangents
(4 cases out of 5 millions trials)

Lines tangent to 4 objects

Lines tangent to 4 triangles

Lower bound: 62
Idea: Transform each blue line into a very thin triangle

Upper bound: 162 (naive: $4 \cdot 3^4 = 324$)

Visibility skeleton of polyhedra

Free segments tangent to 4 among $n$ triangles

**Complexity:**

- Worst case: $\Theta(n^4)$
- Experimental: $\Theta(n^{2.4})$ (but for small $n$) [Dur99]

**Algorithms:**

- Enumeration $\Theta(n^5)$ [DDP99]
  + heuristics $\sim \Theta(n^{2.5})$ observed
- Double sweep $O((n^3 + h) \log n)$ [DDP02]

**Implementations:**

- Robustness issues $\sim$
- No robust implementation
Visibility skeleton of $k$ polytopes

Worst-case size of the visibility skeleton of structured scenes

$n$ triangles organized into $k$ convex polytopes

Disjoint polytopes in generic position $O(n^2 k^2)$ [EGHZ07]

Non-disjoint polytopes in arbitrary position: $\Theta(n^2 k^2)$

[Brönnimann, Devillers, Dujmovic, Everett, Glisse, Goaoc, Lazard, Na, Whitesides, SIAM J. on Comput., 2007]
Worst-case size of the visibility skeleton of structured scenes

\( n \) triangles organized into \( k \) convex polytopes

Disjoint polytopes in generic position \( O(n^2 k^2) \)[EGHZ07]

Non-disjoint polytopes in arbitrary position: \( \Theta(n^2 k^2) \)

Non-disjoint & arbitrary \( \sim \) realistic scenes

Disjoint/non-disjoint: contrast with the 2D case

Visibility skeleton of $k$ polytopes

Idea: Maintain the bitangents in a sweep-plane + amortization
Visibility skeleton of $k$ polytopes

Idea: Maintain the bitangents in a sweep-plane $+$ amortization

[Brönnimann, Devillers, Dujmovic, Everett, Glisse, Goaoc, Lazard, Na, Whitesides, SIAM J. on Comput., 2007]
Visibility skeleton of $k$ polytopes

Idea: Maintain the bitangents in a sweep-plane + amortization

[Brönnimann, Devillers, Dujmovic, Everett, Glisse, Goaoc, Lazard, Na, Whitesides, SIAM J. on Comput., 2007]
Visibility skeleton of $\kappa$ polytopes

**Idea:** Maintain the bitangents in a sweep-plane + amortization

**Lemma:** The number of lines transversal to edge $e$ and tangent to $P, Q, R$ is linear

Visibility skeleton of $\kappa$ polytopes

Idea: Maintain the bitangents in a sweep-plane + amortization

Lemma: The number of lines transversal to edge $e$ and tangent to $P, Q, R$ is linear

Proof:
Number of pairs of bitangents that become aligned
$\leq$ number of pairs of bitangents that share a vertex

Disjoint case: 4 bitangents $\leadsto$ 8 pairs of bitangents per plane; $O(n)$ events

Non-disjoint case: $O(n)$ pairs of bitangents per plane; $O(n)$ events

Amortization $\leadsto O(n)$ pairs

Visibility skeleton of $k$ polytopes

**Idea:** Maintain the bitangents in a sweep-plane + amortization

**Lemma:** The number of lines transversal to edge $e$ and tangent to $P, Q, R$ is linear

**Proof:**
Number of pairs of bitangents that become aligned
\[ \leq \text{number of pairs of bitangents that share a vertex} \]

**Disjoint case:** 4 bitangents $\leadsto$ 8 pairs of bitangents per plane; $O(n)$ events

**Non-disjoint case:** $O(n)$ pairs of bitangents per plane; $O(n)$ events

Amortization $\leadsto O(n)$ pairs

Visibility skeleton of $k$ polytopes

Idea: Maintain the bitangents in a sweep-plane + amortization

Lemma: The number of lines transversal to edge $e$ and tangent to $P, Q, R$ is linear

Proof:
Number of pairs of bitangents that become aligned $\leq$ number of pairs of bitangents that share a vertex

Disjoint case: 4 bitangents $\leadsto 8$ pairs of bitangents per plane; $O(n)$ events

Non-disjoint case: $O(n)$ pairs of bitangents per plane; $O(n)$ events

Amortization $\leadsto O(n)$ pairs

$\leadsto O(n^2k^2 \log n)$ plane-sweep algorithm

Visibility skeleton of $k$ polytopes

**$\beta$-Implementation** (L. Zhang)

Disjoint polytopes

General position (no predicate evaluate to zero)

Filtered exact computation

(CGAL interval arithmetic + CORE exact number type)

Predicate of degree up to 168 [ELLRZ 07]

(ordering two sweep planes through two 4-lines transversals)
Visibility skeleton of $k$ polytopes

**β-Implementation** (L. Zhang)

- Disjoint polytopes
- General position (no predicate evaluate to zero)
- Filtered exact computation
  - (CGAL interval arithmetic + CORE exact number type)
- Predicate of degree up to 168 [ELLRZ 07]
  - (ordering two sweep planes through two 4-lines transversals)

Size of the visibility skeleton in practice?
Efficient computation of shadow boundaries?
Experiments

Generate $k$ uniformly distributed disjoint unit spheres in a spherical universe of density $\mu$

Uniformly generate $n/k$ points on each sphere surface

largest data set:
120 polytopes
10,000 edges

$\mu = 0.3$

[Zhang, Everett, Lazard, Weibel, Whitesides ESA 08]
Visibility skeleton of $k$ polytopes

Number of skeleton vertices: $c \mu k \sqrt{nk}$

$\mu = 0.01$
$\mu = 0.05$
$\mu = 0.3$

$\frac{n}{k} = 6$
$\frac{n}{k} = 42$
$\frac{n}{k} = 84$

$6 \times 10^5$

[Zhang, Everett, Lazard, Weibel, Whitesides ESA 08]

$14 \times 10^4$
Experiment results
The number of vertices in the visibility skeleton of \( k \) random disjoint polytopes of complexity \( n \) is experimentally:

\[
c_{\mu} \sqrt{n} k = c_{\mu} k^2 \sqrt{n/k} \quad (c_{\mu} < 5)
\]

[Zhang, Everett, Lazard, Weibel, Whitesides ESA 08]
Visibility skeleton of \( k \) polytopes

**Experiment results**

The number of vertices in the visibility skeleton of \( k \) random disjoint polytopes of complexity \( n \) is experimentally:

\[
c_\mu \ k \sqrt{n\ k} = c_\mu \ k^2 \sqrt{n/k} \quad (c_\mu < 5)
\]

**Analysis**

Fix the polytope size \( n/k \) complexity is \( \Theta(k^2) \)

**Theory hints** \( k \): \( \Theta(k) \) for randomly distributed unit balls [DDE+ 03]

but scene size is too small...

[Zhang, Everett, Lazard, Weibel, Whitesides ESA 08]
Visibility skeleton of $k$ polytopes

Experiment results
The number of vertices in the visibility skeleton of $k$ random disjoint polytopes of complexity $n$ is experimentally: $c_\mu k \sqrt{n} k = c_\mu k^2 \sqrt{n/k}$ ($c_\mu < 5$)

Analysis
Fix number of polytopes $k$ complexity is $\Theta(\sqrt{n})$
Polyhedron of size $m$ approximating a surface
On average over all viewpoints the silhouette is of size $O(\sqrt{m})$

For spheres [Kettner, Welzl 97]
Arbitrary surfaces [Glisse, Lazard 08]

[Zhang, Everett, Lazard, Weibel, Whitesides ESA 08]
Visibility skeleton of $k$ polytopes

**Experiment results**

The number of vertices in the visibility skeleton of $k$ random disjoint polytopes of complexity $n$ is experimentally: $c_\mu k \sqrt{n/k} = c_\mu k^2 \sqrt{n/k}$ ($c_\mu < 5$)

**Conclusions**

Visibility skeleton $\rightarrow$ Sub-quadratic in the number of polytopes

$\rightarrow$ Sub-linear in the number of triangles

Running time of the sweep algorithm is

$c'_\mu n \sqrt{n} k \log k$ seconds, $c'_\mu < 3 \cdot 10^{-4}$

Much better than the worst case bounds

[Zhang, Everett, Lazard, Weibel, Whitesides ESA 08]
Worst-case size of the umbra in structured scenes

Umbra in a plane: cells of an arrangement of conics

Surprising bounds:

Segment light source

2 triangles
\(\sim\) up to 4 conn. components

2 polytopes
\(\sim\) \(\Theta(n)\) conn. components

\(n\)-gon light source

\(k\) polytopes
\(\sim\) \(\Omega(n^2k^3 + nk^5), O(n^3k^3)\)

[Demouth, Devillers, Everett, Lazard, Glisse, Seidel, Symp. on Comp. Geom., 2007]
Shadow boundaries

Shadow boundary computations [Demouth 08]
Future work

Improvements to the algorithm and implementation for computing the visibility skeleton

General polyhedra
Predicates [Devillers et al. 08]
Degeneracies
Heuristics

Applications to shadow computations and visibility queries
Acknowledgments

Thanks to collaborators:

Borcea, Bronimann, Demouth, Devillers, Dujmovic, Everett, Goaoc, Glisse, Lenhart, Na, Petitjean, Seidel, Sottile, Weibel, Whitesides, Zhang

(Rider Univ.; Poly Univ. NY; INRIA Sophia-Anipolis; CNRS Grenoble; Williams College, Williamstown MA; Soongsil University, Seoul; Saarland University, Saarbrücken; Texas A&M; McGill University, Montréal)
### Bounds on max. free line segments

Number of (cc. of) max. free segments tangent to

<table>
<thead>
<tr>
<th></th>
<th>Worst case</th>
<th>Expected</th>
<th>Observed (random setting)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 lines</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 segments</td>
<td>4 [05]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 triangles</td>
<td>$62 \leq ? \leq 162$ [07]</td>
<td></td>
<td>$\leq 40$ [07]</td>
</tr>
<tr>
<td>4 $\in k$ polytopes</td>
<td>$\Theta(n^2k^2)$ [07]</td>
<td>$O(k^2)$ ($\frac{n}{k} = c$)</td>
<td>$\Theta(k \sqrt{nk})$ [08]</td>
</tr>
<tr>
<td>4 $\in k$ unit balls</td>
<td>$\Theta(n^4)$ [07]</td>
<td>$\Theta(n)$ [03]</td>
<td></td>
</tr>
</tbody>
</table>

#### Algorithms for max free segments tangent to $4 \in k$ polytopes

- $O(n^2k^2 \log n)$ [07]
- $O((n^3 + h) \log n)$ [02]
- $\Theta(n^5)$ [97]
- $\Theta(nk \sqrt{n})$ [08]
- $\Theta(n^{2.5})$ [97]