Excess covariance in financial returns: empirical data and theoretical models

Matteo Marsili
Abdus Salam International Centre for Theoretical Physics

G. Raffaelli (UNIPOL Bologna)
B. Ponsot (CFM Paris)
P. Pin (AS ICTP, Trieste)
M. Anufriev (CeNDEF, Amsterdam)
G. Bottazzi (S. Anna Pisa)

EU-NEST project COMPLEXMARKETS
what are markets for?

(individual optimum) \times N \neq \text{global optimum}

• markets allocate optimally resources
  (It is not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard to their own interest. A. Smith)

• markets incorporate efficiently available information in prices (Fama)
  statistical regularities correspond to what agents are not (yet) doing

• markets allow individuals to cope with uncertainty and reduce risk
  statistical features show what traders are doing
Outline

• Data:
  Why are stock prices so correlated?
  Why are correlations so volatile?

• A phenomenological model

• A micro-economic model
correlation between assets

\[ \text{Cov}_{GE,PFE} = E[\Delta x_{GE} \Delta x_{PFE}] \]

- **N** stocks
- **T** = window size
- **t_0** = initial time
- **\Delta t** = time scale (1 day)
- **\tau** = time shift (=0 here)
Can you really understand market dynamics by looking at a single price?

• Translation invariance:
  
  \[ p_i \rightarrow \lambda p_i, \quad \Leftrightarrow \quad x_i = \log p_i \rightarrow x_i + x_0 \ \forall \text{assets } i \]

  • center of mass - market mode (herding/non-informed trades)
  • relative coordinates - ex-market returns (informed trades)

• Time-scale invariance of correlations of ex-market returns

(see C. Borghesi, MM. S. Micciche PRE 2007)
More precisely:

\[
\begin{align*}
\text{log prices} & \quad x_i(t) = \log p_i(t) \\
\text{log returns} & \quad \Delta x_i(t) = x_i(t + \Delta t) - x_i(t) \\
\text{avg return} & \quad r_i(t) = \mu \sum_{t' \leq t} (1 - \mu)^{t-t'} x_i(t') \\
\text{covariance} & \quad \text{Cov}_{i,j}(t) = \mu \sum_{t' \leq t} (1 - \mu)^{t-t'} [\Delta x_i(t') - r_i(t)][\Delta x_j(t') - r_j(t)] \\
\text{volatility} & \quad \text{Vol}_i(t) = \text{Cov}_{i,i}(t) = \mu \sum_{t' \leq t} (1 - \mu)^{t-t'} [\Delta x_i(t') - r_i(t)]^2 \\
\text{correlation} & \quad \text{Corr}_{i,j}(t) = \frac{\text{Cov}_{i,j}(t)}{\sqrt{\text{Vol}_i(t)\text{Vol}_j(t)}} \\
\end{align*}
\]

$\Delta t = 1 \text{ day}$

$i, j = 1, \ldots, N$
The market mode

Histogram of eigenvalues of the covariance matrix of daily returns of N=41 stocks of NYSE on T=6910 days (1980-2007)

Same for correlation matrix (e.g. S&P500, Laloux et al.)
How do correlations form?
The Epps effect: correlation grows with $\Delta t$

Information is aggregated faster
today than in the past

in bigger companies

(J. Kwapien, S. Drozdz, J. Speth, Toth, Kertesz, Eisler)

... and what about the structure?

FIG. 2: Largest eigenvalue $\Lambda/N$, divided by the number of assets, of the matrix $A_\tau$ as a function of $\tau$ for NYSE, LSE and PB (full symbols). Ratio $\lambda/\Lambda$ of the second largest to the largest eigenvalue of $A_\tau$, as a function of $\tau$ (open symbols).
Time-horizon invariant structure without center of mass

Assets in the same cluster follow the same trajectory across time-scales when center of mass is removed.
The dynamics of covariance

\[ \text{Cov}_{i,j}(t) = \mu \sum_{t'=1}^{t} (1-\mu)^{t-t'} [x_i(t') - r_i(t)][x_j(t') - r_j(t)] \]

\[ r_i(t) = \mu \sum_{t'=1}^{t} (1-\mu)^{t-t'} x_i(t') \]

moving average on window of size 1/\( \mu \)

Sliding window with exponential average
The dynamics of the market mode
(largest eigenvalue of Cov)

(see also Drozdz et al.)

Dow

tsx

Distribution of fluctuations of $\Lambda$

Dow
Dax
Tsx
Asx
N=41 stocks of NYSE
(from yahoo.finance.com)
Where do correlations come from?

- extend GARCH to dynamical conditional correlation models (Engle, Bauwens et al.)
- excess comovement from correlated demand (Greenwood) or index participation (Barberis et al.)
- aggregate liquidity shocks (Brunetti)
- large portfolio manager in illiquid market (1 risky asset - 1 riskless - Bank)
- ...

Where do correlations come from?

• Portfolio investment:
  agents spread investment across stocks to minimize risk
  (i.e. avoid correlations)
  In doing this, they invest in a correlated way in the market → they create correlations

\[
\hat{C} = \hat{B} + \hat{F}(\hat{C}) + \hat{\Omega}
\]

  Economics  Finance  Noise

• Simple models describing this feedback
Phenomenological approach
Multi-asset markets as many “particle” interacting systems

\[ x_i(t) = \log p_i(t) \]

\[ i = 1, \ldots, N \]

\[ (N = \# \text{ assets}) \]

- Prices ≠ correlated random walks

✦ “Bound state”

✦ Collective motion
Phenomenological approach
(as little discipline as possible)

whatever $\Delta x$ could depend on

$$\Delta \vec{x}(t) = \vec{x}(t + 1) - \vec{x}(t) = \vec{F} \left( \vec{x}(t), \partial_t \vec{x}(t), \partial_t^2 \vec{x}(t), ..., \vec{a}(t'), \partial_t \vec{a}(t), ..., \vec{b}(t'), ... \right)$$

$$= \vec{F}(0, 0, ..., t) + \frac{\partial \vec{F}}{\partial \vec{x}} \bigg|_{0, 0, ...} \vec{x}(t) + \frac{\partial \vec{F}}{\partial (\partial_t \vec{x})} \bigg|_{0, 0, ...} \partial_t \vec{x}(t) + ...$$

1- small fluctuations $\rightarrow$ expand in powers of argument
2- high frequency $\rightarrow$ expand in time derivatives
3- eliminate terms which cannot appear
4- study the simplest non-trivial model
5- add complication
Dirac’s bra-kets

- Vectors
  \[ |x\rangle = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \quad \langle x| = (x_1, \ldots, x_N) \]

- Scalar product
  \[ \langle x|y\rangle = \sum_{i=1}^{N} x_i y_i \]

- Direct product (matrix)
  \[ |x\rangle\langle y| = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_N \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_N \\ \vdots & \vdots & \ddots & \vdots \\ x_N y_1 & x_N y_2 & \cdots & x_N y_N \end{pmatrix} \]

- Basis
  \[ |k\rangle, \quad k = 1, \ldots, N, \quad \langle k|j\rangle = N \delta_{k,j}, \quad \hat{I} = \frac{1}{N} \sum_{k=1}^{N} |k\rangle\langle k| \]
  \[ |v\rangle = \frac{1}{N} \sum_{k=1}^{N} v^{(k)} |k\rangle, \quad v^{(k)} = \langle k|v\rangle \]
Optimal Portfolios

- Problem: Invest $|z\rangle \rightarrow$ stochastic return $= \langle \Delta x|z\rangle$
  - expected return $= \langle r|z\rangle = R$, $|r\rangle = E[|\Delta x\rangle ]$
  - wealth $= \langle 1|z\rangle = W$, $|1\rangle =(1,...,1)$

so as to minimize risk $\Sigma(|z\rangle )$

- Solution (if $\Sigma(|z\rangle )=\text{Var}(\langle \Delta x|z\rangle ))$:

$$
|z^{*}\rangle = \arg \min_{|z\rangle,\lambda,\nu} \left[ \frac{1}{2} \langle z|\hat{C}|z\rangle - \lambda(\langle r|z\rangle - R) - \nu(\langle 1|z\rangle - W) \right]
$$

- Note:
  no impact on market, unique solution.
  What if many traders invest in this same way?
  Will this have some impact?
A generic phenomenological model:

\[ \Delta |x_t\rangle = |x_{t+1}\rangle - |x_t\rangle \]
\[ = |F(t, r_t, z_t^{(1)}, \ldots, z_t^{(K)}, \Delta r_t, \Delta z_t^{(1)}, \ldots, \Delta z_t^{(K)})\rangle \]
\[ = |F(t, 0, 0, \ldots)\rangle + \]
\[ \sum_{\ell=1}^{K} \left[ \frac{\partial |F\rangle}{\partial \langle z^{(\ell)}\rangle} \bigg|_0 |z^{(\ell)}_t\rangle + \frac{\partial |F\rangle}{\partial \Delta \langle z^{(\ell)}\rangle} \bigg|_0 \Delta |z^{(\ell)}_t\rangle + \ldots \right] + \ldots \]
\[ = |\alpha_t\rangle + \beta_t |r_t\rangle + \tilde{\beta}_t \Delta |r_t\rangle + \ldots + \sum_{\ell=1}^{K} \left[ \nu_t^{(\ell)} |z^{(\ell)}_t\rangle + \tilde{\nu}_t^{(\ell)} \Delta |z^{(\ell)}_t\rangle + \ldots \right] + \ldots \]

K components of optimal portfolios with parameters \( R^k, W^k, \mu^k \)

low frequency expansion + r, z^k, small \( \rightarrow \) power expansion

Fundamentalists + speculators
(= noise traders)

Chartists, trend followers

Risk managers
The simplest model:
Closed dynamical model, self-generated fluctuations/correlations

- $|x_{t+1}\rangle = |x_t\rangle + |b_t\rangle + v_t |z_t\rangle$

  $|b_t\rangle$ = “bare” returns

  $E[|b_t\rangle] = \bar{b}|1\rangle + \sigma|2\rangle,$

  $E[|b_t\rangle\langle b_t'|] = B\hat{I}\delta_{t,t'}$

- $v_t$ = portfolio investment rate
  $E[v_t] = \bar{v}, \quad \text{Var}[v_t] = \Delta$

- Where
  $|z_t\rangle = \arg\min_{|z\rangle,\lambda,\nu} \left[ \frac{1}{2} \langle z|\hat{C}_t|z\rangle - \lambda(\langle r_t|z\rangle - R) - \nu(\langle 1|z\rangle - W) \right]$

- Average return and correlation matrix ($\mu \sim 1/T_{\text{average}}$)
  $|r_{t+1}\rangle = (1-\mu) |r_t\rangle + \mu \left[ |x_{t+1}\rangle - |x_t\rangle \right]$

  $\mathcal{C}_{t+1} = (1-\mu) \mathcal{C}_t + \mu |\delta x_t\rangle \langle \delta x_t|$

  $|\delta x_t\rangle = |x_t\rangle - |x_{t-1}\rangle - 1$
Numerical simulations

Dynamic instability as $W \to W^*$
...and close to $W^*$
Theory: low frequency limit $\mu \to 0$

- for $\mu \to 0$
  
  $C_t, |r_t\rangle$ independent of $t \to |z_t\rangle$ independent of $t$, $\Delta|z_t\rangle = 0$

- $\Lambda = B + \mathbf{N} \Delta f(\mathbf{R}/\mathbf{N}, \mathbf{W}/\mathbf{N}, \nu)$

- Self-consistent equations
  
  $\to$ phase transition at $W^*$ to dynamically unstable phase

- Small $\mu$ expansion
  
  $\to$ scaling

\[
\frac{\delta \Lambda}{\Lambda} \sim \sqrt{\frac{\mu}{W^* - W}}
\]
Instability in the general model:

\[
\frac{1}{4} \delta b^2 - (\bar{b} + \bar{w}) \bar{w} + (1 - \alpha) \bar{r} - \frac{1}{N} \bar{Z}_\perp^2 \geq 0
\]

1. the phase transition is robust
   - for any risk measures of agents
   - independent of higher order derivative terms
   - noise filtering cannot help

2. the market is less stable
   - the larger the volume of trading
   - the smaller the return demanded
     (i.e. the more agents are risk averse!)
   - the stronger are trend followers
   - the larger or less diversified “bare” returns (∼ dividends)
   - the more correlated are stocks a priori

\[
\mu \to 0 \text{ limit: } \bar{r} = \bar{b} + \alpha \bar{r} + \sum_{\ell} \epsilon^\ell \bar{z}^\ell
\]

\[
\bar{w} = \frac{1}{N} \sum_{\ell} \epsilon^\ell W^\ell
\]

\[
\bar{r} = \frac{1}{N} \sum_{\ell} \epsilon^\ell R^\ell
\]

\[
\bar{Z}_\perp = \sum_{\ell} \epsilon^\ell \bar{z}^\ell
\]

\[
\bar{z}_\perp^\ell \cdot \bar{b} = 0
\]
What does it really depend on?

- Impact of portfolio strategies changes one constraints from hyper-plane to hyper-sphere ($\varepsilon \neq 0$)
- no risk free asset
Back to real markets

1. Fit model to real market data
   1. compute likelihood
   2. maximize $\rightarrow$ parameters
   3. markets are close to instability

2. Compare market’s and model’s behaviors
   1. what picture does the model provide?
   2. does that picture agree with market data?
The model:
Switching among the two principal directions

\[ |b_t\rangle = \bar{b}|1\rangle + \sigma|2\rangle \]

NYSE 41 stocks: Overlap on (1,1,...)
The model’s picture: Covariance, volatility and correlation

\[ W \ll W_c \]

\[ W = W_c \]

\[ W \gg W_c \]
The real picture: N=41 stocks of NYSE (from yahoo.finance.com)
Summary I

• log p translation invariance suggests different dynamics of center of mass and relative coordinates
• Complex dynamics of global correlations
• Simple model for the feedback of correlations through portfolio strategies
  – exhibit market mode
  – develops complex dynamics at a phase transition
  – suggests markets are close to a phase transition
• Real markets seem to be more complex
Economics inspired approach

(+ M. Anufriev, G. Bottazzi, P. Pin)
What micro-economics?

• CARA models (Chiarella, Dieci, He)
  – demand independent of wealth
  – stationary equilibrium
    • price growth exogenous
    • prices can go negative

• CRRA models
  – demand proportional to wealth
  – dynamic equilibrium
    • price growth endogenous
    • positive prices
Multi-asset market: 1 agent

- N assets (1 unit) + risk free asset (return $r_f$)
- discrete time; $t=1, 2, ...$
- homogeneous investors (=1 agent): wealth $w_t$
  portfolio
  \[ |x_t⟩ = (x_1^t, \ldots, x_N^t) \]
- market clearing:
  \[ |p_t⟩ = w_t |x_t⟩ \]
- wealth dynamics
  \[ w_{t+1} = w_t [1 + r_f + ⟨x_t| r_{t+1}⟩ + ⟨x_t| δ_{t+1}⟩] \]
- excess returns and dividend yields (or news arrival process)
  \[ r_{t+1}^a = \frac{p_{t+1}^a - p_t^a}{p_t^a} - r_f \quad |δ_t⟩ ∈ \mathcal{N}(|d⟩, \hat{D}) \]
- how is $|x⟩$ chosen by the agent?
Equilibrium

- Time indep. portfolio \( |x_{t+1}\rangle = |x_t\rangle = |x\rangle \)

- Returns \( |r_{t+1}\rangle = \frac{\langle x|\delta_{t+1}\rangle}{1 - \langle 1|x\rangle} |1\rangle \) (ex-dividend) Prices grow at the same rate (apart from dividends, there is a only one game)

- Expected return \( |c\rangle = \frac{\langle x|d\rangle}{1 - \langle 1|x\rangle} |1\rangle + |d\rangle \) Price volatility is \( \sim N \) times higher than dividend volatility

- Covariance matrix

\[
\hat{C} = \hat{D} + \frac{\langle x|\hat{D}|x\rangle}{(1 - \langle 1|x\rangle)^2} |1\rangle\langle 1| + \frac{|1\rangle\langle x|\hat{D} + \hat{D}|x\rangle\langle 1|}{1 - \langle 1|x\rangle}
\]

Large eigenvector \( \sim N \) in Cov and Corr

\textit{Note:}

\[\text{singularity when } \langle x|1\rangle \to 1\] Separation “Theorem”: equilibrium is independent of \( r_f \)
“Strong” CAPM

- the excess return of each asset a is given by

\[ r_t^a = \frac{\langle \delta_t | x \rangle}{1 - \langle x | 1 \rangle} = \frac{1}{\langle x | 1 \rangle} \langle r_t | x \rangle \]

- valid for all t, not just for expected values

- same beta’s

\[ \beta^a = \frac{1}{\langle x | 1 \rangle} = \frac{\text{Cov}(r_t^a, \langle r_t | x \rangle)}{\text{Var}(\langle r_t | x \rangle)} \]

\[ = \gamma + \frac{\langle 1 | \hat{D}^{-1} | d \rangle}{\langle 1 | \hat{D}^{-1} | d \rangle} \]

(41 stocks in NYSE)
Mean–variance strategies: CRRA $\gamma$

The agents are CRRA and choose a portfolio taking into account expected first and second moments.

They compute mean vector

$$|c_t\rangle = E_t [|r_{t+1}\rangle + |\delta_{t+1}\rangle]$$

and variance–covariance matrix

$$\hat{C}_t = Cov_t \left( |r_{t+1}\rangle + |\delta_{t+1}\rangle, |r_{t+1}\rangle + |\delta_{t+1}\rangle \right)$$

They choose the portfolio

$$|x\rangle = \frac{1}{\gamma} \hat{C}_t^{-1} |c_t\rangle,$$

where $\gamma$ is a positive parameter of risk aversion.

Equilibrium:

$$|x\rangle = \frac{1}{\gamma + \langle 1 | \hat{D}^{-1} | d \rangle} \hat{D}^{-1} |d\rangle$$

No short selling:

$$x_i > 0, \quad 1 - \langle x | 1 \rangle > 0$$
**Mean–variance strategies:**
**portfolio managers**

The agents are risk–neutral portfolio managers with a constraint

They also compute mean vector \( |c_t\rangle \) and variance–covariance matrix \( \hat{C}_t \)

They fix an excess return \( \rho = R = \langle x|c\rangle \), minimizing risk \( \Sigma = \frac{1}{2} \langle x|\hat{C}|x\rangle \)

Solution is \( |x\rangle = R \frac{\hat{C}^{-1}|c\rangle}{\langle c|\hat{C}^{-1}|c\rangle} \)

Analogy: \( \gamma \) as \( \frac{R}{\langle c|\hat{C}^{-1}|c\rangle} \)

Equilibrium:

\[
|x\rangle = \frac{1}{N \left( R\bar{d} + \bar{d}^2 + \sigma^2 \right)} R|d\rangle
\]

No short selling also here!
How does the market depend on agent’s behavior?

We can normalize expected return and variance by the risky portfolio

\[ \rho = \frac{\langle x|c \rangle}{1 - \langle x|1 \rangle} + r_f, \quad \text{and} \quad \Sigma = \frac{\sqrt{\langle x|\hat{C}|x \rangle}}{1 - \langle x|1 \rangle}. \]

Explicit curves for optimal portfolios are:

- **R portfolio managers:** $\Sigma = \sqrt{\frac{D}{N(d^2 + \sigma^2)}} \left( \rho - r_f \right)$

- **CRRA $\gamma$:** $\Sigma = \sqrt{\frac{D}{N(d^2 + \sigma^2)}} \sqrt{1 + \frac{d^2 + \sigma^2}{d(\rho - r_f)}} \left( \rho - r_f \right)$
Learning to be a mean-variance investor

\( \mu \) updating and \( \nu \) inertia

\[
\begin{align*}
i) \quad |x_t\rangle & = (1 - \nu)|x_{t-1}\rangle + \frac{\nu}{\gamma} \hat{C}^{-1}_{t-1}|c_{t-1}\rangle \\

ii) \quad r^i_t & = \frac{\langle x_{t-1}|\delta_t\rangle x^i_t + (1 + r_f)(1 - \langle x_{t-1}|1\rangle)x^i_t}{(1 - \langle x_t|1\rangle)x^i_{t-1}} - 1 - r_f \\

iii) \quad |c_t\rangle & = (1 - \mu)|c_{t-1}\rangle + \mu E_\delta \left[ |r_{t-1}\rangle + |\delta_{t-1}\rangle \right] \\

iv) \quad \hat{C}_t & = (1 - \mu)\hat{C}_{t-1} + \mu \text{Cov}(|r_{t-1}\rangle + |\delta_{t-1}\rangle) \\

\hat{D} = D\hat{I}
\end{align*}
\]

The expectation over \( |\delta_t\rangle \) could be simply equal to last observation
Assuming $\mu \to 0$, $\nu \to 0$ (randomness $\to 0$),

and $\frac{\nu}{\mu} \equiv \lambda$, we can take continuum limit ($\tau \equiv \mu t$)

\[
\frac{d\langle x \rangle}{d\tau} = -\lambda \langle x \rangle + \frac{\lambda}{\gamma} \hat{C}^{-1} \langle c \rangle \\
\frac{d\langle c \rangle}{d\tau} = -\langle c \rangle + \frac{\langle x | d \rangle}{1 - \langle x | 1 \rangle} \langle 1 \rangle + \langle d \rangle \\
\frac{d\hat{C}}{d\tau} = -\hat{C} + \frac{\langle x | \hat{D} | x \rangle}{(1 - \langle x | 1 \rangle)^2} \langle 1 \rangle \langle 1 \rangle + \frac{\langle 1 \rangle \langle x | \hat{D} + \hat{D} | x \rangle \langle 1 \rangle}{1 - \langle x | 1 \rangle} + \hat{D}
\]

$\hat{D} = D \hat{I}$: we consider only projections on $|1\rangle$ and $|2\rangle$

(as in equilibrium)

We have a 7–dimensional system: its Jacobian has all negative eigenvalues

For the $R$ portfolio managers case we change $\frac{d\langle x \rangle}{d\tau}$: it is still stable
Evolutionary (in)-stability

A part of the population changes slightly from $|x\rangle$ to $|x + \epsilon\rangle$
(still in the $|1\rangle - |2\rangle$ plane)

We can compute the gradient of a variation that brings higher expected wealth growth

$$
\partial E \left[ \log \left( \frac{w_{t+1,\epsilon}}{w_{t,\epsilon}} \right) \right] \approx \frac{E \left[ |\delta_{t+1}\rangle + r_{t+1} |1\rangle \right]}{1 + r_f + r_{t+1} + \langle \epsilon | \delta_{t+1}\rangle + r_{t+1} \langle \epsilon | 1\rangle}
$$

Since yields $|\delta_{t+1}\rangle$ are all positive:

Evolutionary pressure brings towards instability
Many (M) agents, many (N) assets

- Portfolios $|x_{t,a}\rangle$, $a = 1, \ldots, M$
- Wealths
  \[
  w_{t+1,a} = w_{t,a} \left[ (1 + r_f) (1 - \langle x_{t,a} | 1 \rangle + \langle x_{t,a} | \delta_t \rangle) + w_{t,a} \langle x_{t,a} | R_{t+1} \rangle \right]
  \]
  gain from risk free asset and dividends capital gain

- ex-dividend returns
  \[
  R_{t+1}^i = \frac{p_{t+1}^i}{p_t^i} = \frac{\sum_a w_{t+1,a} x_{t+1,a}^i}{\sum_a w_{t,a} x_{t,a}^i}
  \]

- non-linear coupled dynamics
Numerical simulations

i) more than one agent survives!
ii) non-trivial fluctuations when $M > N$
Summary

• Multi-asset model:
  – structure of correlations (market mode)
  – Single agent
    • picture of how agents shape the market
    • stability of learning dynamics
    • evolutionary pressure towards instability
    • stable correlations: no non-trivial fluctuations
  – Many agents
    • stable correlations with few types
    • non-trivial fluctuations with many types (numerical)
Thanks

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marsili@ictp.it