Dielectrics & Polarization

Some notes from Lecture #8 on Friday February 22.

These notes do not stand on their own; they should be “consumed” in combination with my lecture.

We discussed twice before that non-conductors can become polarized due to an external electric field. The field can induce dipoles in atoms and molecules; the stronger the field, the stronger the degree of polarization. We call these materials dielectrics\(^1\). We exclude all ideal conductors in which charge flows in response to the field. What I will discuss now is notoriously confusing.

Symbols without arrows represent magnitudes.

Take a plate capacitor (conducting plates). We connect it to a battery (of voltage \(V\)) and consequently charge the plates with the same amount of charge but of opposite sign. I will call the charge on each plate \(Q_{\text{free}}\), and the resulting surface charge density \(\sigma_{\text{free}}\) (this is \(Q_{\text{free}}\) divided by the surface area of one plate), and I call the electric field inside the capacitor due to this charge, \(E_{\text{free}}\). Notice, Giancoli, on page 625 mentions the words “free charge”, but uses the symbols \(Q\) and \(\sigma\) (without the subscript “free”).

\[
\begin{align*}
E &= 0 \\
\text{d} & \text{ } \\
E_{\text{free}} & \text{ } \\
-\sigma_{\text{free}} & \text{ } \\
+\sigma_{\text{free}} & \text{ }
\end{align*}
\]

\[
E_{\text{free}} = \frac{\sigma_{\text{free}}}{\varepsilon_0}
\]

After the capacitor is charged up, I disconnect the battery that supplied the potential difference \(V\) between the plates. The charge on the plates is now “trapped”. We now shove a dielectric between the plates. It produces two layers of induced charge on the dielectric (remember the “+ -” transparency shown in lectures). I will call this induced surface charge density \(\sigma_{\text{ind}}\); it is the result of polarization of the dielectric under influence of the electric field. The induced charge is often called “bound charge”, as opposed to the “free charge”.

\[
\begin{align*}
E &= 0 \\
-\sigma_{\text{ind}} & \text{ } \\
+\sigma_{\text{ind}} & \text{ } \\
E_{\text{free}} & \text{ } \\
E_{\text{ind}} & \text{ } \\
+\sigma_{\text{free}} & \text{ } \\
-\sigma_{\text{free}} & \text{ }
\end{align*}
\]

\(^1\) The molecules of some dielectrics have an intrinsic dipole moment even in the absence of an external electric field. When they are “exposed” to an \(E\) field, they try to line up with the field. The degree of success depends on the strength of the applied field. They are included here.
The induced charges produce “their own” electric field, \( \vec{E}_{\text{ind}} \), which opposes the field \( \vec{E}_{\text{free}} \).

\[
E_{\text{ind}} = \frac{\sigma_{\text{ind}}}{\varepsilon_0}
\]  

(2)

and the resulting net field is

\[
\vec{E} = \vec{E}_{\text{free}} + \vec{E}_{\text{ind}}
\]  

(3)

The magnitude of \( E \) is thus less than that of \( E_{\text{free}} \) (the electric field has become weaker).

\[
E = E_{\text{free}} - E_{\text{ind}}
\]  

(4)

Under “normal” circumstances, \( E_{\text{ind}} \propto E_{\text{free}} \); thus \( E_{\text{ind}} = bE_{\text{free}} \), where \( b \) is a constant which depends only on the dielectric substance. Thus, \( \vec{E}_{\text{ind}} = -b\vec{E}_{\text{free}} \). We substitute this in eq. (3), and find

\[
\vec{E} = (1 - b)\vec{E}_{\text{free}}
\]

The constant \((1 - b)\) is called \(1/\kappa\) (\(\kappa\) is the dielectric constant; it depends only on the material that is inserted between the plates). Thus,

\[
\vec{E} = \frac{\vec{E}_{\text{free}}}{\kappa}
\]  

(5)

This is a key equation.

In our experiment where the free charge was trapped (the battery was disconnected before we inserted the dielectric), the dielectric material lowered the field strength by a factor \(\kappa\). Note, the potential difference between the plates, \(V\), must go down by the same factor as \(d\) remains unchanged (\(V\) always equals the product of the total \(E\) field \((E_{\text{tot}})\) between the plates and the distance \(d\) between the plates). However, if I keep the battery connected while I stick the dielectric in, the potential difference between the plates remains unchanged; thus \(E\) can not go down, therefore \(Q_{\text{free}}\), and thus \(E_{\text{free}}\), must go up, consistent with eq. (5); the additional charge that will then flow to the plates will, of course, be delivered by the battery.

As an example, if I use glass, with \(\kappa \approx 5\) (see Table 24.1, page 622 of Giancoli), the \(E\) field will be reduced by a factor of \(\approx 5\). For water, \(\kappa \approx 80\) (this is enormous), the \(E\) field will be reduced by a factor of 80 (for ice at \(-40^\circ C\), \(\kappa \approx 100\)).

In vacuum, per definition, \(\kappa = 1.0000000\) exactly. For most gases, \(\kappa\) is only a “hair” larger than 1.000 which means that \(b\) is very small, and for most applications in air, unless stated otherwise, we use for \(\kappa\) exactly 1.

It follows from the above relation between \(\kappa\) and \(b\) that

\[
E_{\text{ind}} = \left(1 - \frac{1}{\kappa}\right)E_{\text{free}}
\]  

(6)
Since $E_{ind} = \frac{\sigma_{ind}}{\varepsilon_0}$, and $E_{free} = \frac{\sigma_{free}}{\varepsilon_0}$ (see eqs. 1 and 2), it follows that

$$\sigma_{ind} = b\sigma_{free} = \left(1 - \frac{1}{\kappa}\right)\sigma_{free}$$  

(7)

**Eq. 7 gives us the magnitude of the induced surface charge density**, $\sigma_{ind}$, if $\sigma_{free}$, and $\kappa$ are known. Keep in mind that the induced charge is negative (*positive*) near the plate where the free charge is positive (*negative*); see the figure above and below. If you feel the need to express the difference in sign, you could add a minus sign to eq. (7), but I don’t advise that. In all my equations $\sigma_{ind}$ and $\sigma_{free}$ represent the magnitude of the charge; that is why my figures indicate $-\sigma_{ind}$ and $-\sigma_{free}$ where needed.

**Notice** that for very high values of $\kappa$ ($b \approx 1$), $\sigma_{ind} \approx \sigma_{free}$, and the net surface charge density at either plate is then $\approx 0$, and there is **almost no field between the plates!**

**Does Gauss’s law still hold? Of course!** Observe the pill box below; the flat top and bottom (each with area $A$) are parallel to the plates.

The integral of the electric flux over the entire surface of the pill box (*closed surface integral*):

$$\oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} \sum Q_{enclosed}$$  

(8)

and $Q_{enc} = A(+\sigma_{free} - \sigma_{ind})$. The pill box contains the “free” charge on the conducting plate and the “induced” charge on the surface of the dielectric; both have to be taken into account.

The electric field is assumed to be zero outside the capacitor, and uniform in the dielectric. At the curved cylindrical surface, $\vec{E}$ and $d\vec{A}$ are perpendicular to each other, but at the bottom flat surface of the pill box (area $A$), $\vec{E}$ and $d\vec{A}$ have the same direction. Thus, eq. (8) becomes:

$$EA = \frac{1}{\varepsilon_0} (\sigma_{free} - \sigma_{ind}) A$$  

(9)

Using eq. (7), I can eliminate $\sigma_{ind}$, and find:

$$E = \frac{\sigma_{free}}{\kappa \varepsilon_0} = \frac{E_{free}}{\kappa}$$  

(10)
This is exactly what we found above (eq. 5).

It simplifies matters to modify eq. (8) and replace the \textit{enclosed} charge (which includes both the induced and the free charge) by the \textit{free charge} only:

\[
\oint \vec{E} \cdot d\vec{A} = \frac{1}{\kappa \varepsilon_0} \sum Q_{\text{enclosed,free}}
\]  

(11)

which is \textbf{Gauss’s law for dielectrics}.

Notice that eqs. (5), (10) and (11) do not explicitly contain the induced surface charge density \( \sigma_{\text{ind}} \) (i.e. the polarization charge); they only contain the surface charge density on the conducting plate (i.e., the “free” charge).

\textbf{The \( \kappa \) takes the induced charge \textit{automatically} into account.}

\textit{Walter Lewin}