Sinusoidal Steady State
We now understand the why of:

Today, look at response of networks to sinusoidal drive.

Sinusoids important because signals can be represented as a sum of sinusoids. Response to sinusoids of various frequencies -- aka frequency response -- tells us a lot about the system.
Motivation

For motivation, consider our old friend, the amplifier:

Observe $v_o$ amplitude as the frequency of the input $v_i$ changes. Notice it decreases with frequency.

Also observe $v_o$ shift as frequency changes (phase).

Need to study behavior of networks for sinusoidal drive.
Sinusoidal Response of RC Network

Example:

\[ v_I(t) = V_i \cos \omega t \quad \text{for } t \geq 0 \quad (V_i \text{ real}) \]

\[ v_I(t) = 0 \quad \text{for } t < 0 \]

\[ v_C(0) = 0 \quad \text{for } t = 0 \]
Our Approach

Example:

\[ v_I \]

\[ R \]

\[ i_C + \]

\[ v_C - \]

Determine \( v_C(t) \)

Indulge me!

Our Approach

This lecture

11:00

11:20

12:00

Next lecture

Usual approach

easy

agony

sneaky approach

very sneaky

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Let's use the usual approach...

1. Set up DE.
2. Find $v_p$.
3. Find $v_H$.
4. $v_C = v_p + v_H$, solve for unknowns using initial conditions.
Usual approach...

1. Set up DE

\[ RC \frac{dv_C}{dt} + v_C = v_I \]

\[ = V_i \cos \omega t \]

That was easy!
Find \( v_p \)

\[
RC \frac{dv_p}{dt} + v_p = V_i \cos \omega t
\]

First try: \( v_p = A \) → nope

Second try: \( v_p = A \cos \omega t \) → nope

Third try: \( v_p = A \cos(\omega t + \phi) \)

\[
- RCA \omega \sin(\omega t + \phi) + A \cos(\omega t + \phi) = V_i \cos \omega t
\]

\[
- RCA \omega \sin \omega t \cos \phi - RCA \omega \cos \omega t \sin \phi + A \cos \omega t \cos \phi - A \sin \omega t \sin \phi = V_i \cos \omega t
\]

gasp!

works, but trig nightmare!
Let's get sneaky!

Find particular solution to another input...

\[ RC \frac{dv_{PS}}{dt} + v_{PS} = v_{IS} = V_i e^{st} \]

Try solution \( v_{PS} = V_p e^{st} \)

\[ RC \frac{dV_p e^{st}}{dt} + V_p e^{st} = V_i e^{st} \]

\[ sRC V_p e^{st} + V_p e^{st} = V_i e^{st} \]

\[ (sRC + 1)V_p = V_i \]

\[ V_p = \frac{V_i}{1 + sRC} \]

Thus, \( v_{PS} = \frac{V_i}{1 + sRC} \cdot e^{st} \)

is particular solution to \( V_i e^{st} \)

\[ \frac{V_i}{1 + j\omega RC} \cdot e^{j\omega t} \]

solution for \( V_i e^{j\omega t} \)

where we replace \( s = j\omega \)

complex amplitude

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Fourth try to find $v_P$ using the sneaky approach

**Fact 1:** Finding the response to $V_i e^{j\omega t}$ was easy.

**Fact 2:**

$v_I = V_i \cos \omega t$

$= \text{real}[V_i e^{j\omega t}] = \text{real}[v_{IS}]$

from Euler relation,

$e^{j\omega t} = \cos \omega t + j \sin \omega t$

an inverse superposition argument, assuming system is real, linear.
Fourth try to find $v_p$. 

so, \[
\begin{align*}
\text{complex} \\
v_p &= Re[v_{PS}] = Re[V_pe^{j\omega t}] \\
&= Re\left[\frac{V_i}{1 + j\omega RC} \cdot e^{j\omega t}\right] \\
&= Re\left[\frac{V_i(1 - j\omega RC)}{1 + \omega^2 R^2 C^2} \cdot e^{j\omega t}\right] \\
&= Re\left[\frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cdot e^{j\phi} e^{j\omega t}\right], \tan \phi = -\omega RC \\
&= Re\left[\frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cdot e^{j(\omega t + \phi)}\right] \\
v_p &= \frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cdot \cos(\omega t + \phi)
\end{align*}
\]

Recall, $v_p$ is particular response to $V_i \cos \omega t$. 

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3) Find $v_H$

Recall, $v_H = A e^{\frac{-t}{RC}}$
Find total solution

\[ v_C = V_P + v_H \]

\[ v_C = \frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\omega t + \phi) + A e^{\frac{-t}{RC}} \]

where \( \phi = \tan^{-1}(-\omega RC) \)

Given \( v_C(0) = 0 \) for \( t = 0 \)

so,

\[ A = -\frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\phi) \]

Done! Phew!
We are usually interested only in the particular solution for sinusoids, i.e. after transients have died.

Notice when $t \to \infty$, $v_C \to v_P$ as $e^{-\frac{t}{RC}} \to 0$

$$v_C = \frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\omega t + \phi) + A e^{-\frac{t}{RC}}$$

where $\phi = \tan^{-1}(-\omega RC)$

$$A = -\frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\phi)$$

Described as

**SSS: Sinusoidal Steady State**
All information about SSS is contained in $V_p$, the complex amplitude!

Recall

$$V_p = \frac{V_i}{1 + j\omega RC}$$

Steps 3, 4 were a waste of time!

$$\frac{V_p}{V_i} = \frac{1}{1 + j\omega RC} \quad e^{j\phi} \quad \text{where} \quad \phi = \tan^{-1} - \omega RC$$

**magnitude**

$$\left| \frac{V_p}{V_i} \right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

**phase $\phi$**

$$\angle \frac{V_p}{V_i} = -\tan^{-1} \omega RC$$
Visualizing the process of finding the particular solution $v_p$

$V_i \cos \omega t$

\[ D.E. + \text{nightmare trig.} \]

\[ |v_p| \cos[\omega t + \angle v_p] \]

\[ V_p e^{j\omega t} \]

\[ \text{algebraic equation} + \text{complex algebra} \]

\[ \text{sneak in} \]

\[ V_i e^{j\omega t} \]

\[ \text{drive} \]

\[ \text{drive} \]

\[ \text{take real part} \]

\[ \text{sneaky path!} \]
Magnitude Plot

transfer function

\[ H(j\omega) = \frac{V_p}{V_i} \]

\[ \left| \frac{V_p}{V_i} \right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \]

From demo: explains \( v_o \) fall off for high frequencies!
Phase Plot

\[ \phi = \tan^{-1} - \omega RC \]

\[ \phi = \angle \frac{V_p}{V_i} \]

\[ \omega = \frac{1}{RC} \]