Time Dependent Stick Breaking Processes

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Introduction

We are interested in flexibly modelling distributions that change over time.

We observe data $y_1, y_2, \ldots, y_n$ at times $t_1, t_2, \ldots, t_n$.

A popular nonparametric approach assumes a time-dependent infinite mixture model

$$ p(y_i | t_i) = \int N(y_i | \mu, \sigma^2) \, dF_{t_i}(\mu, \sigma^2) = \sum_{j=1}^{\infty} w_j(t_i) N(y_i | \mu_j, \sigma_j^2) $$
The definition of $F_t$ often generalizes generic constructions of standard nonparametric process such as the Dirichlet process.

For example,

- the Polya urn representation of the Dirichlet process.
- normalizing time-dependent random measures.
- the stick-breaking construction of the Dirichlet process.
Stick Breaking Processes

A random distribution $F$ is a stick-breaking process if

$$F = \sum_{j=1}^{\infty} w_j \delta_{\theta_j}$$

where

$$w_j = V_j \prod_{k<j} (1 - V_k)$$

and

- $V_1, V_2, V_3, \ldots$ are independent and $V_j \sim \text{Be}(a_j, b_j)$.
- $\theta_1, \theta_2, \theta_3, \ldots \sim i.i.d. H$. 
Examples

- The Dirichlet process with concentration parameter $M$ arises if $V_j \sim \text{Be}(1, M)$.
- The Poisson-Dirichlet (Pitman-Yor) process arises if $V_j \sim \text{Be}(1 - b, M + bj)$. 
Random Walks on Discrete Distributions

Suppose that $y_t$ is a real-value random variable then we can define a random walk in two (equivalent) ways:

$$y_{t+1} = \mathcal{N}(y_t, \sigma^2) \text{ or } y_{t+1} = y_t + \epsilon_t, \epsilon_t \sim \mathcal{N}(0, \sigma^2).$$

With random distribution we could write

$$F_{t+1} \sim \text{DP}(M, F_t) \text{ or } F_{t+1} = V_t F_t + (1 - V_t) \epsilon_t.$$  

These constructions are not equivalent since $F_t$ is a discrete distribution.

Under the first process $F_t$ tends to a point mass as $t \to \infty$. 
If $\epsilon_t$ is a distribution with a single atom at $\theta_t$ then

$$F_t = \sum_{j=-\infty}^{t} \delta_{\theta_j} V_j \prod_{k=j+1}^{t} (1 - V_k)$$

which is stick-breaking “backwards in time”.

This type of process would be rather inflexible. Allowing $V$’s to arrive according to a Poisson process and applying the stick-breaking process increases flexibility.
DPAR process

\[ F_t = \sum_{j=1}^{\infty} p_j(t) \delta_{\theta_j} \]

where

\[ p_j(t) = \begin{cases} 0 & \tau_j > t \\ V_j \prod_{k_1 \leq \tau_k < \tau_j < t}(1 - V_k) & \tau_j < t \end{cases} \]

for

- \( V_1, V_2, V_3, \ldots \) \( i.i.d. \) \( \text{Be}(1, M) \).
- \( \tau_1, \tau_2, \tau_3, \ldots \) follow a Poisson process with intensity \( \lambda \) on \( (-\infty, \infty) \).
- \( \theta_1, \theta_2, \theta_3, \ldots \) \( i.i.d. \) \( H \).
DPAR process

$t = 0$

![Graph showing initial state of DPAR process](image)

$t = 1$

![Graph showing updated state of DPAR process](image)

Increment

![Graph showing increment in DPAR process](image)
DPAR process

$t = 1$

$t = 2$

Increment
DPAR process

\[ t = 2 \]

\[ t = 3 \]

Increment
Properties

- $F_t$ is DP($M, H$) for all $t$.
- The autocorrelation is

$$\text{Corr}(F_t(B), F_{t+s}(B)) = \exp\left\{-\frac{\lambda}{M+1}s\right\}$$
Chinese restaurant representation

We integrate over the atoms which have no point associated with them to give a finite dimensional representation of the process.

Let the active set at time \( r \) be

\[
A(r) = \# \{ j \mid \tau_s < r \leq t_j \}
\]

\[
m_j = \sum_{i=1}^{n} I(s_i = j)
\]

Let \( T_n \) be the subset of our Poisson process which have an observation allocated to them and \( S_n = T_n \cup \{ t_1, \ldots, t_n \} \).
Chinese restaurant representation

\[
(1-C_2)D_2C_1D_1C_0 \quad \quad (1-C_i)D_1C_0 \quad \quad 1-C_0
\]

\[
(1-D_2)C_1D_1C_0 \quad \quad (1-D_i)C_0
\]

\[
D_i = \frac{1 + m_j}{1 + m_j + M + A(\tau_j)}
\]

\[
C_0 = \rho(t_{n+1} - \max\{S_n\}), \quad C_i = \rho^{\frac{M(M+1)}{(M+A(\phi_i))(1+M+A(\phi_i))}}(\tau_i - \tau_{i-1})
\]
If we draw a new value within \((\tau_{l-1}, \tau_l)\) then the new position is 
\[ \tau^* = \tau_l - x \] where \(x\) is exponentially distributed with parameter 
\[ \frac{\lambda M(\tau_l - \tau_{l-1})}{(M+A(\tau_l))(M+A(\tau_l)+1)} \] truncated to the region \((0, \tau_l - \tau_{l-1})\).
Π-AR processes

In the DPAR the distribution of the $V$’s do not depend on their position in the ordering at a given time.

For more general process, such as Poisson-Dirichlet, we need the distribution of $V$ to depend on its position in the ordering. The position is changing over time so we need $V$ to follow a stochastic process.
Π-AR processes

\[ F_t = \sum_{i=1}^{\infty} p_j(t) \delta_{\theta_j} \]

where

\[ p_j(t) = \begin{cases} 
0 & \tau_j > t \\
V_j(t) \prod_{k|\tau_j<\tau_k<t}(1 - V_k(t)) & \tau_j < t
\end{cases} \]

where

- \( V_1(t), V_2(t), V_3(t), \ldots \sim \text{i.i.d. Be}(1, M) \).
- \( \tau_1, \tau_2, \tau_3, \ldots \) follow a Poisson process with intensity \( \lambda \) on \( (-\infty, \infty) \).
- \( \theta_1, \theta_2, \theta_3, \ldots \sim \text{i.i.d. } H. \)
If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are both non-decreasing sequences then the random sequence $V^{(1)}, V^{(1)}, V^{(3)}, \ldots$ defined by

$$V^{(j+1)} = w_j V^{(j)} + (1 - w_j) \epsilon_j$$

where $w_j \sim \text{Be}(a_j + b_j, a_{j+1} + b_{j+1} - a_j - b_j)$ and $\epsilon_j \sim \text{Be}(a_{j+1} - a_j, b_{j+1} - b_j)$ implies that $V^{(j)} \sim \text{Be}(a_j, b_j)$. 
PDAR

We say that \( \{F_t\}_{t=-\infty}^{\infty} \) follows a PDAR\((M, b, \lambda, H)\) if

\[
F_t = \sum_{i=1}^{\infty} p_j(t) \delta_{\theta_j}
\]

where \( \tau_1, \tau_2, \tau_3, \ldots \) follow a homogeneous Poisson process with intensity \( \lambda \) and \( \theta_1, \theta_2, \theta_3, \ldots \) \( i.i.d. \) \( H \) and

\[
p_j(t) = \begin{cases} 
0 & \tau_j > t \\
V_j(t) \prod_{k|\tau_j < \tau_k < t} (1 - V_k(t)) & \tau_j < t
\end{cases}
\]

and

\[
V_j(t) = \epsilon_j \prod_{m=1}^{t} w_{j,m}
\]

where \( \epsilon_j \sim \text{Be}(1 - b, M + b) \), \( w_{j,1} = 1 \) and \( w_{j,m} \sim \text{Be}(1 + M + b(m - 1), b) \).
PDAR

\[ b = 0 \]

\[ b = 0.1 \]

\[ b = 0.2 \]

\[ C = 2 \]

\[ C = 4 \]

\[ C = 8 \]
Computation

Markov chain Monte Carlo are fairly straightforward by extending the Retrospective Sampling methods of Papaspiliopoulos and Roberts (2008).

The breaks of the PDAR process have a product form and the posterior can be sampled using an extension of methods for Matrix Stick Breaking Processes.
Stochastic volatility

Let $p_1, p_2, \ldots, p_T$ be the daily values of a stock index

$$y_t = \log p_{t+1} - \log p_t = \sqrt{h_t} \epsilon_t$$

where $\epsilon_t$ are independently drawn from some returns distribution and the conditional variance $h_t$ is modelled

$$\log h_t \sim N(\delta \log h_{t-1}, \sigma_v^2).$$

where

$$p(\epsilon_t) = \int N(\mu, \sigma^2) \, dF_t(\mu, \sigma^2)$$
Stochastic volatility

Standard and Poors index from 1/1/1980 until 31/12/1988
Stochastic volatility

Fitted return distributions over time
Stochastic volatility

Fitted return distributions at various times
Stochastic volatility

Estimated variance over time
Time-dependent density estimation

We model real (log) per capita GDP over 110 EU regions from 1977 to 1996.
Time-dependent density estimation

Fitted distributions over time
Time-dependent density estimation

Fitted distributions for each year
Time-dependent density estimation
• \( \pi \)-AR process allow us to construct stochastic process on probability measures with a wide-class of stick-breaking processes as marginals.

• DPAR has a “Chinese restaurant”-type representation which may be useful for non-MCMC estimation methods.

• The processes define jump processes on probability measures and are useful for finding change-points.