Min, Max and PTIME Anti-Monotonic Overlap Graph Measures

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Introduction

Single graph mining

Mine frequent patterns in single, typically large graph, e.g.,

- worldwide web graph
- social networks
- biological networks
- citation graphs

Patterns are graphs themselves

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Patterns are graphs themselves

Main question

How define frequency of a graph in a graph?

Not as easy as it seems!

Anti-monotonicity

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Support measure f is anti-monotonic if frequency of superpattern P is never higher than frequency of subpattern p:

$$\forall p, P, G \mid p \subseteq P : f(p, G) \geq f(P, G)$$

$\Rightarrow$ Algorithms rely critically on anti-monotonicity to prune search space

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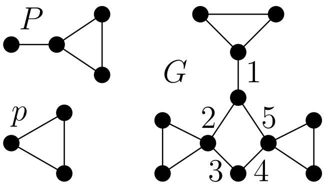

Counting instances does not work!

- $f(p, G) = 3 < f(P, G) = 5$

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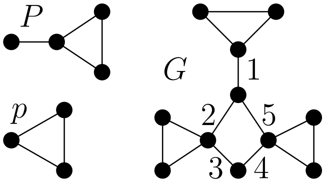

Counting instances does not work!

- $f(p, G) = 3 < f(P, G) = 5$
- frequency P higher than frequency $p \subseteq P$!
- not anti-monotonic

Overlap graphs

Most anti-monotonic support measures in single graph mining are based on overlap graph:

- Undirected graph G_P with instances of pattern P as vertices
- Edge between vertices if correspondig instances overlap in G
- Two types of overlap: edge or vertex overlap

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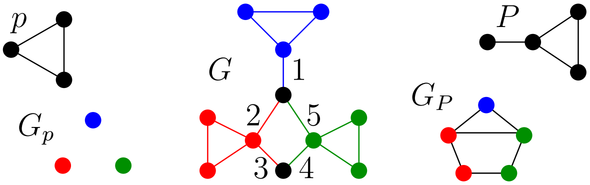
Example for vertex overlap:



Extension to labeled or directed graphs straightforward

Relation between overlap graphs

N. Vanetik, S. E. Shimony, E. Gudes, "Support measures for graph data", Data Min. Knowl. Discov., Vol. 13 (2), 2006, 243–260:

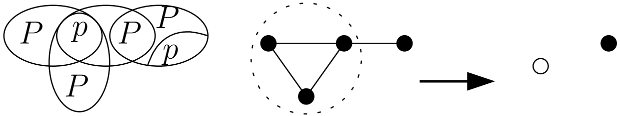
Observations

- Each instance of P contains instance of p
- G_P can be transformed into G_p by three operations:
  1. Clique Contraction (CC)
  2. Vertex Addition (VA)
  3. Edge Removal (ER)

Clique Contraction

Clique Contraction (CC)

contracts clique formed by multiple instances of P sharing single instance of p to single vertex, only overlap with all P-instances of clique remains



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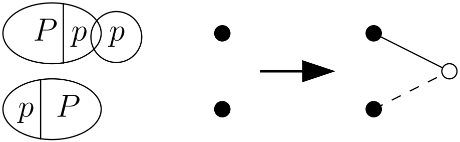
Vertex Addition

Vertex Addition (VA)

adds instance of p not part of instance of P, assumes overlap with all other instances



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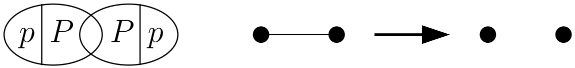
Edge Removal

Edge Removal (ER)

removes redundant edges

- from overlap between instances of P for which contained instances of p do not overlap
- redundant edges introduced by Vertex Addition



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Anti-monotonic overlap graph measures

Overlap graph measure

Support measure f is overlap graph measure
$\iff \forall P, G : f(P, G) = \hat{f}(G_P)$

- Overlap graph measure uses only topology overlap graph
- We call $\hat{f}$ graph measure associated with f

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Theorem (Vanetik, Gudes, Shimony)

Overlap graph measure f is anti-monotonic
$\iff$ *$\hat{f}$ is increasing under CC, VA, ER*

Proof for labeled graphs and edge overlap graph measures

Meaningful measures

Overlap measure f is meaningful if

(1) f anti-monotonic

(2) frequency of k non-overlapping matches is k

$\Rightarrow \hat{f}(\overline{K_k}) = k$ ($\overline{K_k} = k$ isolated vertices)

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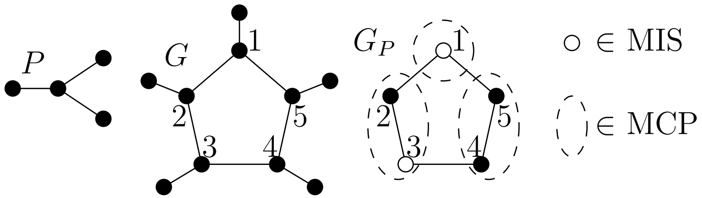

Bounding theorem for meaningful measures

Theorem

For every meaningful measure f it holds that:

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   - $\hat{f}(\overline{K_k}) = k = MIS(G_P)$ and in each step $\hat{f}$ can only increase $\Rightarrow \hat{f}(G_P) \geq MIS(G_P)$

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    - each V_i is clique and can be contracted by CC

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   - $\hat{f}(\overline{K_k}) = k$ and in each step $\hat{f}$ can only increase $\Rightarrow$ $\hat{f}(G_P) \leq k = MCP(G_P)$

Bad News

MIS and MCP are both NP-hard problems in size overlap graph

PTIME computable graph measure

A known PTIME graph measure is the Lovász number θ:

Lovász number

$\theta(G) = \min_A \lambda_{\max}(A)$, where $\lambda_{\max}(A)$ denotes largest eigenvalue of matrix A and minimum is taken over all feasible matrices A such that $A^\top = A$, $A_{ii} = 1$ and $A_{ij} = 1$ if $(i, j) \notin E(G)$.

Theorem (Sandwich Theorem, Lovász)

$$MIS(G) \leq \theta(G) \leq MCP(G)$$

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Theorem

θ is meaningful overlap graph measure

Proof: linear algebra

Generalization of results of Vanetik, Gudes and Shimony

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  2. vertex overlap graphs?
  3. homomorphic or homeomorphic matches?

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- What about
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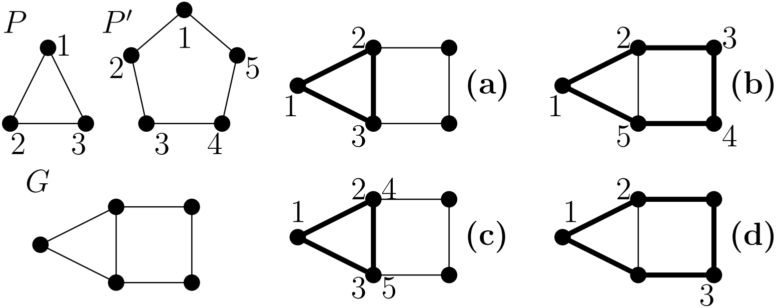
  3. homomorphic or homeomorphic matches?
  4. any combination of the above?

Other types of matches

1. subgraph homomorphism $\cong$ subgraph isomorphism without injectivity
2. subgraph homeomorphism $\cong$ subgraph isomorphism in which edges can be mapped on vertex-disjoint paths



(a) iso of P **(b)** iso of P' **(c)** homo of P' **(d)** homeo of P in G

Extension to other settings

All results in this presentation:

1. f anti-monotonic overlap graph measure
   $\iff$ $\hat{f}$ increasing under CC, VA and ER
2. MIS, MCP, θ are meaningful (anti-monotonic + $\hat{f}(\overline{K_k}) = k$)
3. all meaningful overlap measures f:
   $MIS(G) \leq \hat{f}(G) \leq MCP(G)$

can be extended to all 24 combinations of

1. labeled or unlabeled graphs
2. directed or undirected graphs
3. isomorphic, homomorphic or homeomorphic matches
4. vertex or edge overlap graphs

Proofs are either direct or reductions from one setting to another