Identifying Optimal Sequential Decisions

Vanessa Didelez

Department of Mathematics

University of Bristol

joint work with A. Philip Dawid (Cambridge)

Helsinki, July 2008

Main Messages

To find an optimal sequential decision strategy must allow it to depend on all available information.

Conditions for identifying a decision strategy that depends on all available information are ‘simple’.

Note: we use influence diagrams instead of causal DAGs.

Data Situation

$A_1, \ldots, A_N$ "action" variables $\rightarrow$ can be 'manipulated'

$L_1, \ldots, L_N$ covariates $\rightarrow$ (available) background information

$Y = L_{N+1}$ response variable

all measured over time, L_i before A_i

$\mathbf{A}^{<i} = (A_1, \ldots, A_{i-1})$ past up to before i; $\mathbf{A}^{\leq i}$, $\mathbf{A}^{>i}$ etc. similarly

Example

Consider patients receiving anticoagulant treatment $\Rightarrow$ has to be monitored and adjusted.

$L_i =$ blood test results, other health indicators.

$A_i =$ dose of anticoagulant drug.

Plausibly, 'optimal' dose A_i will be a function of $\mathbf{L}^{\leq i}$ (and poss. $\mathbf{A}^{<i}$).

Strategies

Strategy $\mathbf{s} = (s_1, \ldots, s_N)$ set of functions assigning an action

$a_i = s_i(\mathbf{a}^{<i}, \mathbf{l}^{\leq i})$ to each history $(\mathbf{a}^{<i}, \mathbf{l}^{\leq i})$

(Could be stochastic, then dependence on $\mathbf{a}^{<i}$ relevant.)

Also called: **conditional** / **dynamic** / **adaptive** strategies.

Evaluation

Let $p(\cdot;\mathbf{s})$ be distribution under strategy $\mathbf{s}$.

Let $k(\cdot)$ be a loss function. Want to evaluate $E(k(Y);\mathbf{s})$.

Define

$$f(\mathbf{a}^{\leq j},\mathbf{l}^{\leq i}) := E\{k(Y)|\mathbf{a}^{\leq j},\mathbf{l}^{\leq i};\mathbf{s}\} \quad i=1,\ldots,N; j=i-1,i.$$

Then obtain $f(\emptyset)=E(k(Y);\mathbf{s})$ from $f(\mathbf{a}^{\leq N},\mathbf{l}^{\leq N})$ iteratively by:

$$f(\mathbf{a}^{<i},\mathbf{l}^{\leq i}) = \sum_{a_i} \underbrace{p(a_i|\mathbf{a}^{<i},\mathbf{l}^{\leq i};\mathbf{s})}_{\text{known by s}} \times f(\mathbf{a}^{\leq i},\mathbf{l}^{\leq i})$$

$$f(\mathbf{a}^{<i},\mathbf{l}^{<i}) = \sum_{l_i} p(l_i|\mathbf{a}^{<i},\mathbf{l}^{<i};\mathbf{s}) \times f(\mathbf{a}^{<i},\mathbf{l}^{\leq i}).$$

(cf. extensive form analysis)

Identifiability

Problem: doctors are following their 'gut feeling' (and not a specific strategy) in modifying the dose of anticoagulant drug.

Identifiability: *can* we find the optimal strategy from such data?

In particular: $p(l_i|\mathbf{a}^{<i},\mathbf{l}^{<i};\mathbf{s})$ then not known.

But could *estimate* $p(l_i|\mathbf{a}^{<i},\mathbf{l}^{<i};o)$ under observational regime.

Introduce indicator

$$\sigma = \begin{cases} o, & \text{observational regime} \\ s, & \mathbf{s} \in \mathcal{S} = \text{ set of strategies} \end{cases}$$

Simple Stability

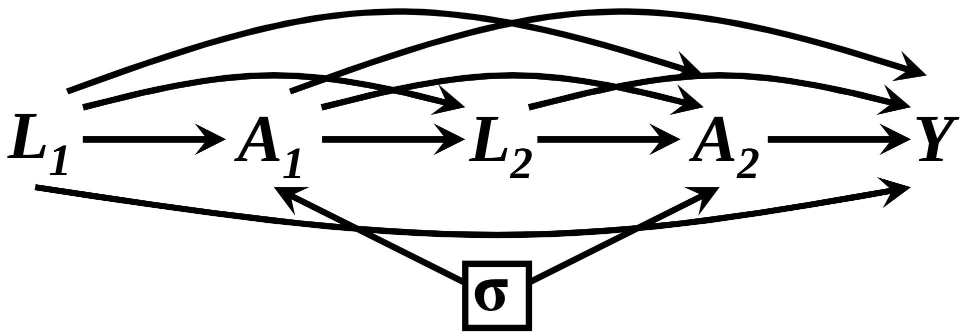
Sufficient for identifiability is

$$p(l_i|\mathbf{a}^{<i}, \mathbf{l}^{<i}; \mathbf{s}) = p(l_i|\mathbf{a}^{<i}, \mathbf{l}^{<i}; o) \quad \text{for all } i = 1, \ldots, N+1$$

or (via intervention indicator)

$$L_i \perp\!\!\!\perp \sigma | (\mathbf{A}^{<i}, \mathbf{L}^{<i}) \quad \text{for all } i = 1, \ldots, N+1$$

Or graphically:

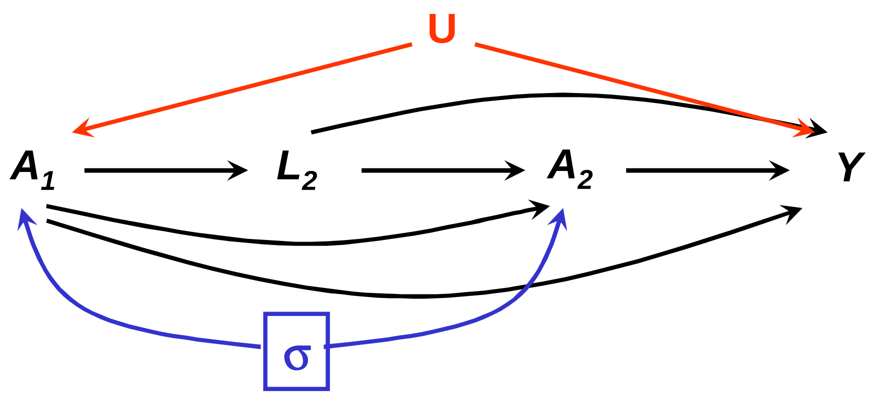


Extended Stability

Might not be able to assess simple stability without taking unobserved variables into account.

$\Rightarrow$ extend covariates $\mathbf{L}$ to include unobserved / hidden variables $\mathbf{U} = (U_1, \ldots, U_N)$ and check if simple stability can be deduced.

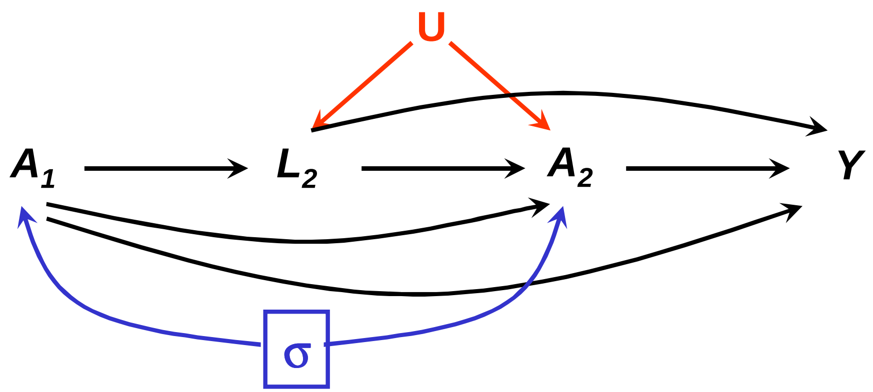
Example 1: particular underlying structure (note: $L_1 = \emptyset$)



Simple stability violated as $Y \not\perp\!\!\!\perp \sigma \mid (A_1, A_2, L_2)$.

Examples

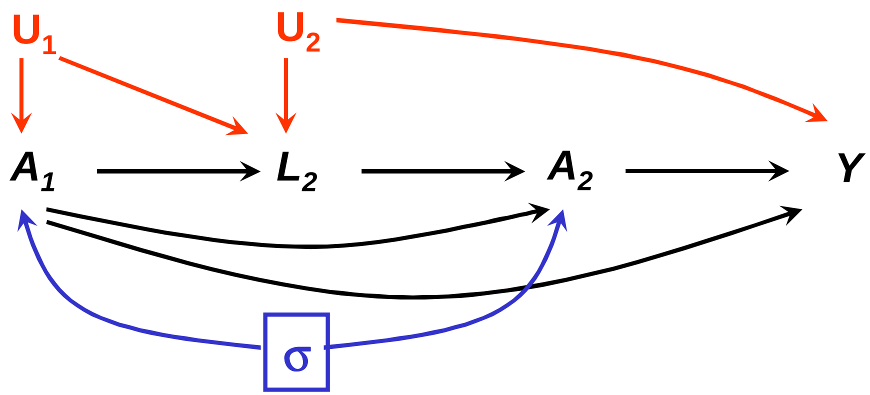
Example 2: different underlying structure



Simple stability satisfied.

Examples

Example 3: another different underlying structure



Simple stability violated: $L_2 \not\perp\!\!\!\perp \sigma \mid A_1$ and $Y \not\perp\!\!\!\perp \sigma \mid (A_1, A_2, L_2)$

Relax Simple Stability?

For given strategy $\mathbf{s}$ can relax conditions for identifiability.

Assume extended stability holds wrt. $\mathbf{A}, \mathbf{L}, \mathbf{U}, Y$, and define 'new' joint distributions $p_i(\mathbf{A}, \mathbf{L}, \mathbf{U}, Y) =$

$$p(\mathbf{A}^{\leq i}, \mathbf{L}^{\leq i}, \mathbf{U}^{\leq i}; o) \times p(\mathbf{A}^{>i}, \mathbf{L}^{>i}, \mathbf{U}^{>i}, Y | \mathbf{A}^{\leq i}, \mathbf{L}^{\leq i}, \mathbf{U}^{\leq i}; \mathbf{s})$$

Relax Simple Stability?

For given strategy $\mathbf{s}$ can relax conditions for identifiability.

Assume extended stability holds wrt. $\mathbf{A}, \mathbf{L}, \mathbf{U}, Y$, and define 'new' joint distributions $p_i(\mathbf{A}, \mathbf{L}, \mathbf{U}, Y) =$

$$\underbrace{p(\mathbf{A}^{\leq i}, \mathbf{L}^{\leq i}, \mathbf{U}^{\leq i}; {\color{red}o})}_{\color{red}\text{obs. for } \leq i} \times p(\mathbf{A}^{>i}, \mathbf{L}^{>i}, \mathbf{U}^{>i}, Y | \mathbf{A}^{\leq i}, \mathbf{L}^{\leq i}, \mathbf{U}^{\leq i}; \mathbf{s})$$

Relax Simple Stability?

For given strategy $\mathbf{s}$ can relax conditions for identifiability.

Assume extended stability holds wrt. $\mathbf{A}, \mathbf{L}, \mathbf{U}, Y$, and define 'new' joint distributions $p_i(\mathbf{A}, \mathbf{L}, \mathbf{U}, Y) =$

$$p(\mathbf{A}^{\leq i}, \mathbf{L}^{\leq i}, \mathbf{U}^{\leq i}; o) \times \underbrace{p(\mathbf{A}^{>i}, \mathbf{L}^{>i}, \mathbf{U}^{>i}, Y | \mathbf{A}^{\leq i}, \mathbf{L}^{\leq i}, \mathbf{U}^{\leq i}; \mathbf{s})}_{\text{strategy for } > i}$$

Relax Simple Stability?

For given strategy $\mathbf{s}$ can relax conditions for identifiability.

Assume extended stability holds wrt. $\mathbf{A}, \mathbf{L}, \mathbf{U}, Y$, and define 'new' joint distributions $p_i(\mathbf{A}, \mathbf{L}, \mathbf{U}, Y) =$

$$p(\mathbf{A}^{\leq i}, \mathbf{L}^{\leq i}, \mathbf{U}^{\leq i}; o) \times p(\mathbf{A}^{>i}, \mathbf{L}^{>i}, \mathbf{U}^{>i}, Y | \mathbf{A}^{\leq i}, \mathbf{L}^{\leq i}, \mathbf{U}^{\leq i}; \mathbf{s})$$

Theorem 1: sufficient condition for identifiability of $\mathbf{s}$ is

$$p_{i-1}(y|\mathbf{a}^{\leq i}, \mathbf{l}^{\leq i}) = p_i(y|\mathbf{a}^{\leq i}, \mathbf{l}^{\leq i}), \qquad i = 1, \ldots, N.$$

(Simple stability implies the above.)

Comments

Theorem 1, in words:
once we know a_i and the observable past variables the distribution of Y does not depend on *how* a_i was generated, when $\mathbf{a}^{<i}$ is observational and $\mathbf{a}^{>i}$ follows the strategy.

Note:
Essentially same as Pearl & Robins (1995) for *unconditional* strategies.

Comments

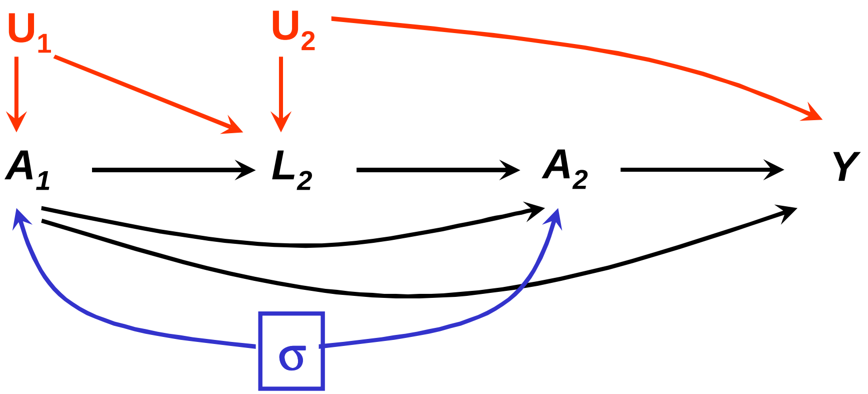
Theorem 1, in words:
once we know a_i and the observable past variables the distribution of Y does not depend on *how* a_i was generated, when $\mathbf{a}^{<i}$ is observational and $\mathbf{a}^{>i}$ follows the strategy.

Graphical check: draw graphs D_i with
— pa(A_k) as under observational regime for $k < i$
— pa(A_k) as under strategy for $k > i$
— pa(A_i) union of both regimes and σ.

$\Rightarrow$ check separation $Y \perp\!\!\!\perp \sigma | (\mathbf{A}^{\leq i}, \mathbf{L}^{\leq i})$ in D_i, $i = 1, \ldots, N$

Example 3 ctd.

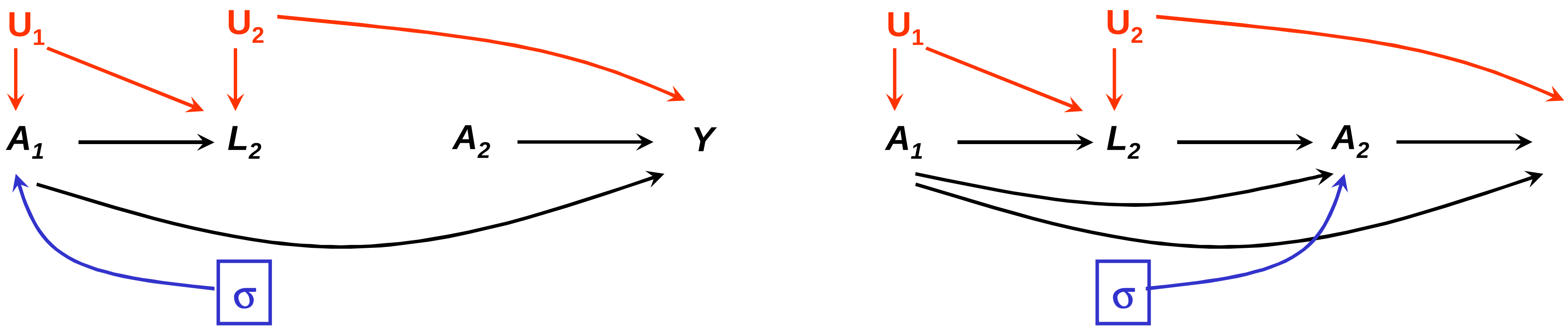
Assumed underlying structure (note: $L_1 = \emptyset$ here)



Now: also assume that s_2 is unconditional, i.e. choice of action A_2 in our strategy does not depend on past observations.

Example 3 ctd.

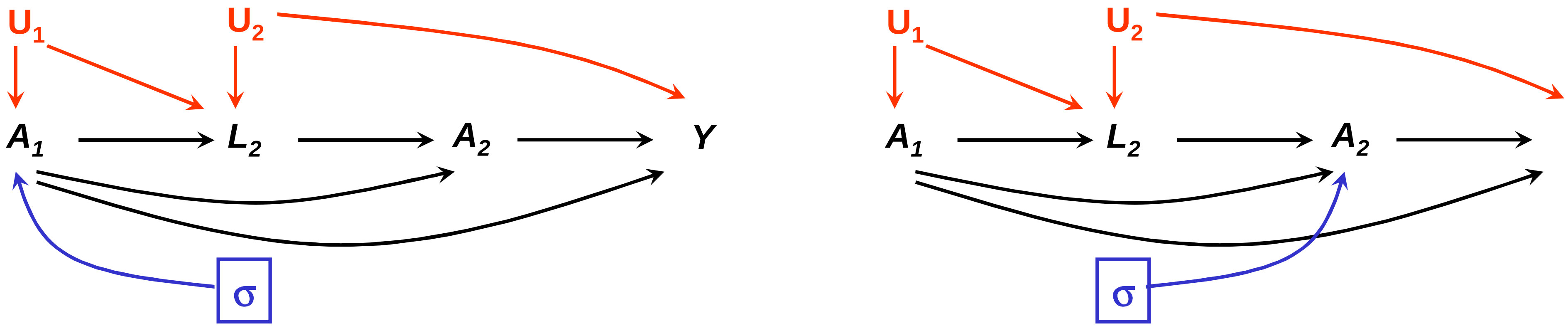
Then D_1 and D_2 are given by



Can see that $Y \perp\!\!\!\perp \sigma | A_1$ in D_1 and $Y \perp\!\!\!\perp \sigma | (A_1, A_2, L_2)$ in D_2.

Example 3 ctd.

However, if s_2 is conditional, i.e. A_2 depends on past observations in our strategy, then D_1 and D_2 are given by



Now $Y \not\perp\!\!\!\perp \sigma | A_1$ in D_1.

This suggests that the 'relaxed' conditions are not so 'relaxed' for conditional interventions.

Result

Assumption 1: $\text{pa}_{\text{s}}(A_i) \subset \text{pa}_o(A_i)$ for all $i = 1, \ldots, N$.

Assumption 2: each $L_1, \ldots, L_N$ is an ancestor of Y in D_0 (as under strategy $\mathbf{s}$), $i = 1, \ldots, N$.

Theorem 2: With these assumptions, if the graphical check of Theorem 1 succeeds then we also have simple stability.

Optimal strategies: Assumption 2 satisfied because
— actions A_i must be allowed to depend on past $\mathbf{L}^{\leq i}$
— and A_i ancestors of Y.

Conclusions and Outlook

- A given strategy s can be identified using simple stability or Theorem 1 as criteria.
  $\Rightarrow$ latter is cumbersome to check.

- When we aim at searching for an optimal strategy, we need to be able to identify strategies that are conditional on all available information.
  $\Rightarrow$ only need to check simple stability criterion.

- For the same graphical structure, an unconditional strategy may be identified while a conditional one is not.