Matching Point Sets with respect to the Earth Mover's Distance

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Today

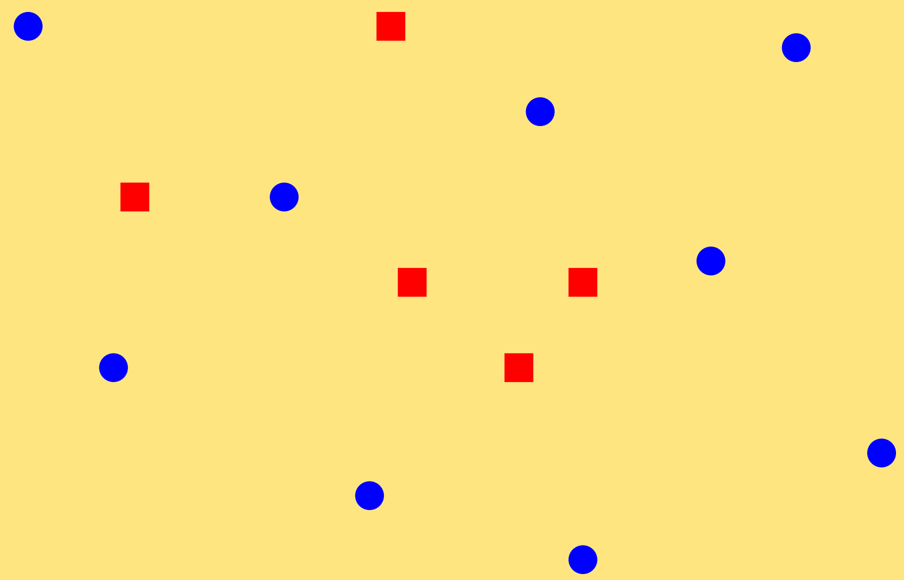
- Earth Mover's Distance.
- EMD for Pattern Matching.
- A glimpse of the results.
- A glimpse of the ideas.

Earth Mover's Distance

n supplies of earth

m demands of earth

unit demand/supply



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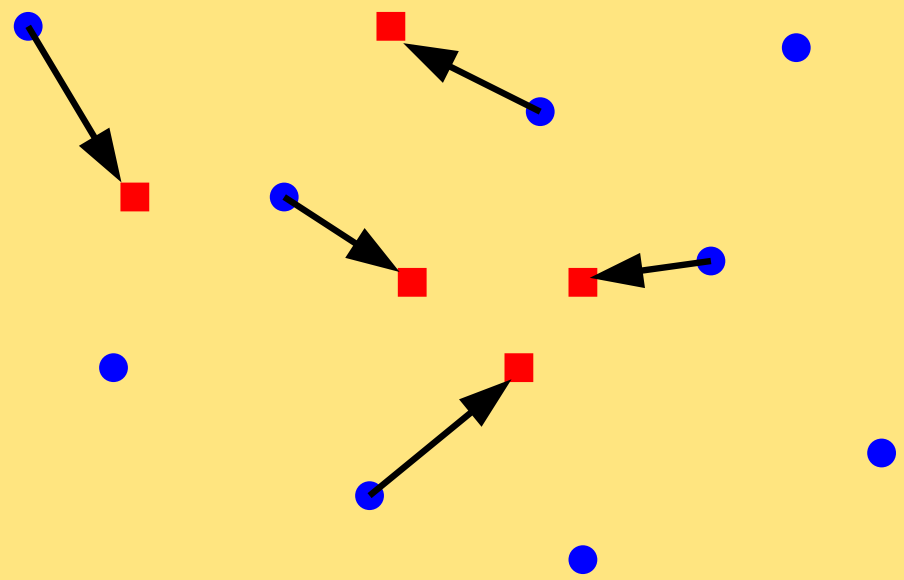

EMD is minimum $\frac{\text{length red-blue matching}}{\min\{n,m\}}$

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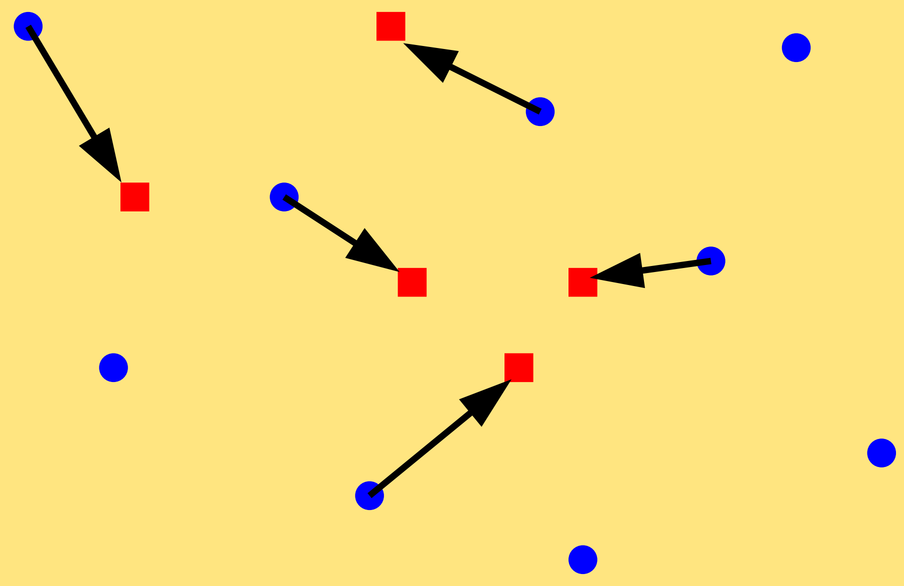

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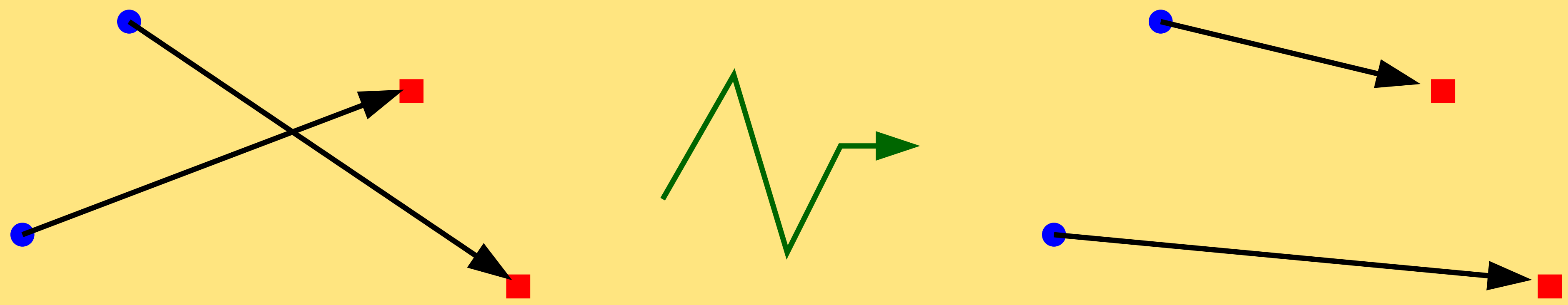
Monge-Kantorovich mass transportation problem

Earth Mover's Distance

Optimal solution has the Monge property:

No two arrows cross



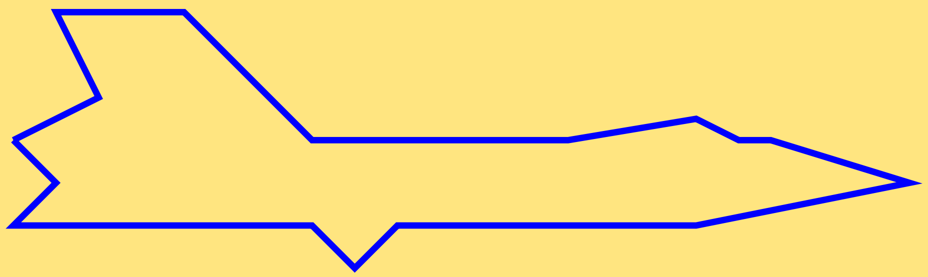
Earth Mover's Distance

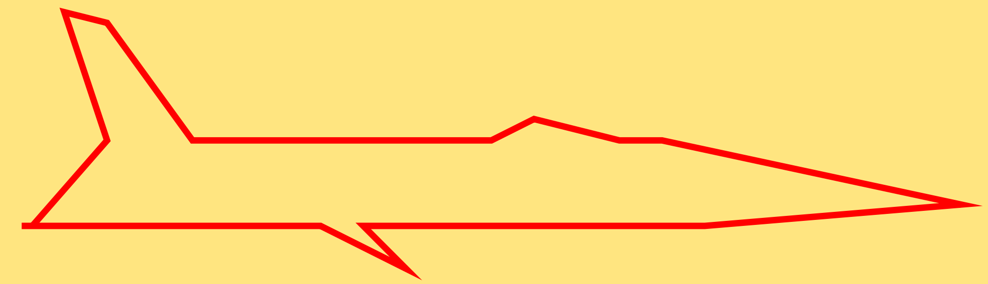
Earth Mover's Distance

- Weighted demands and supplies
- Arbitrary metric spaces
- total demand $=$ total supply (or not)
- min cost max flow problem $\Rightarrow$ solvable by standard techniques

Today: only unit demand/supply

EMD for Shape Matching





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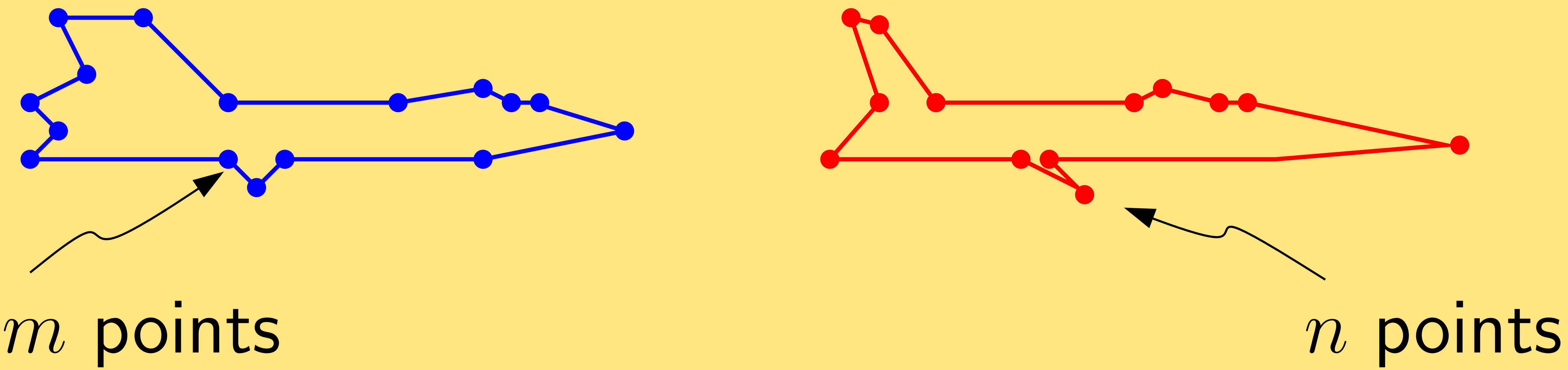

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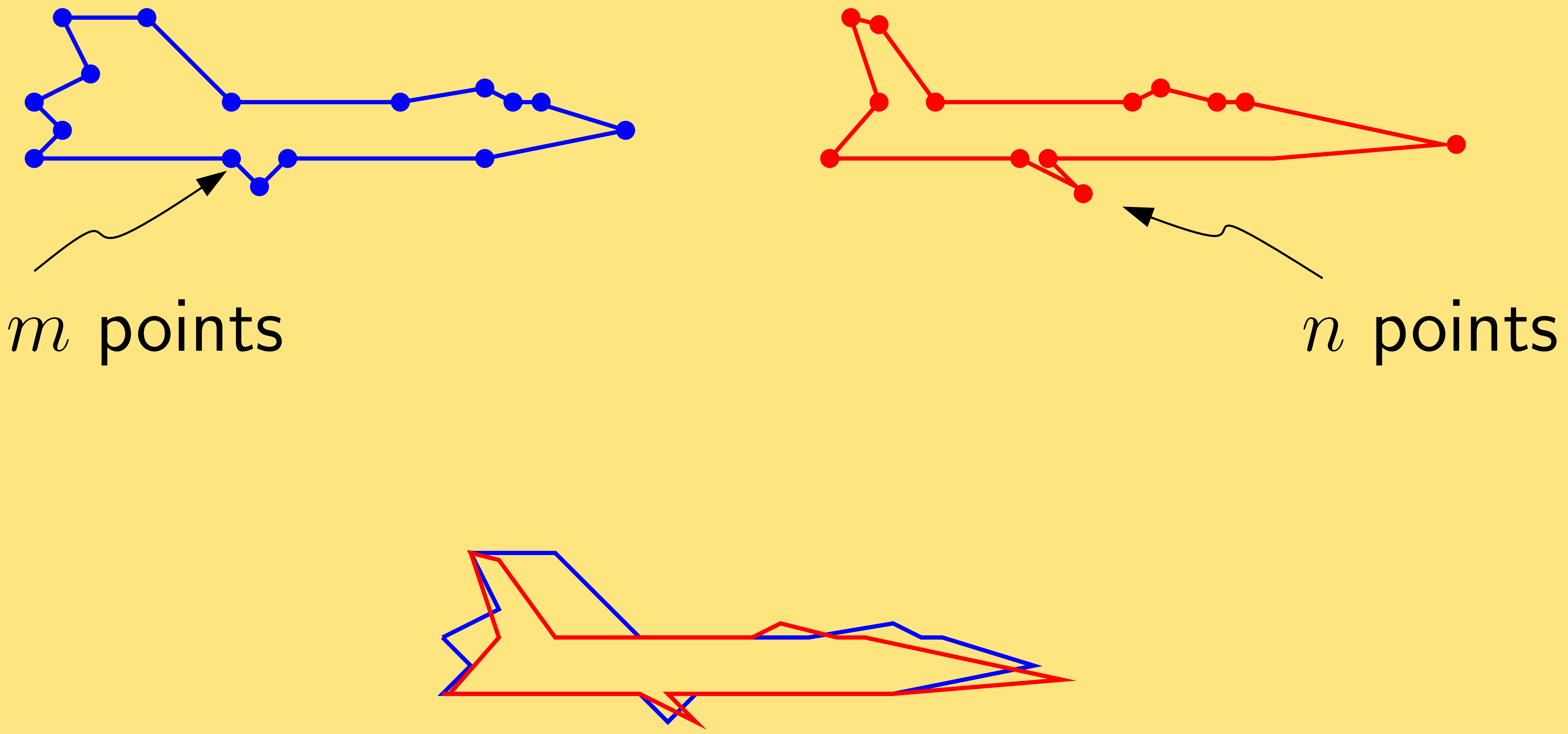

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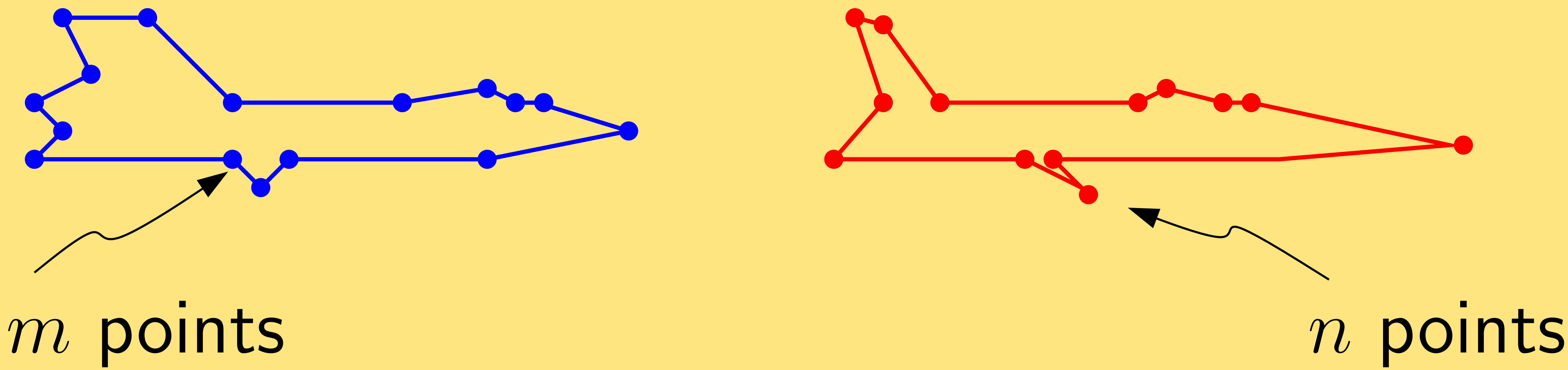
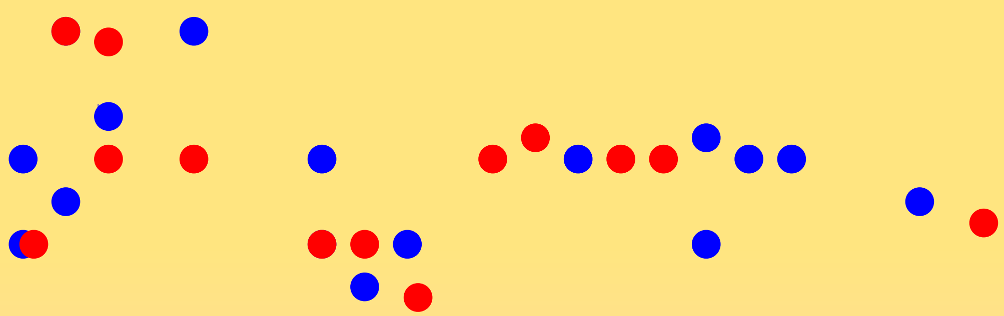
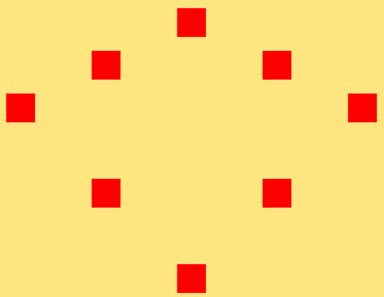




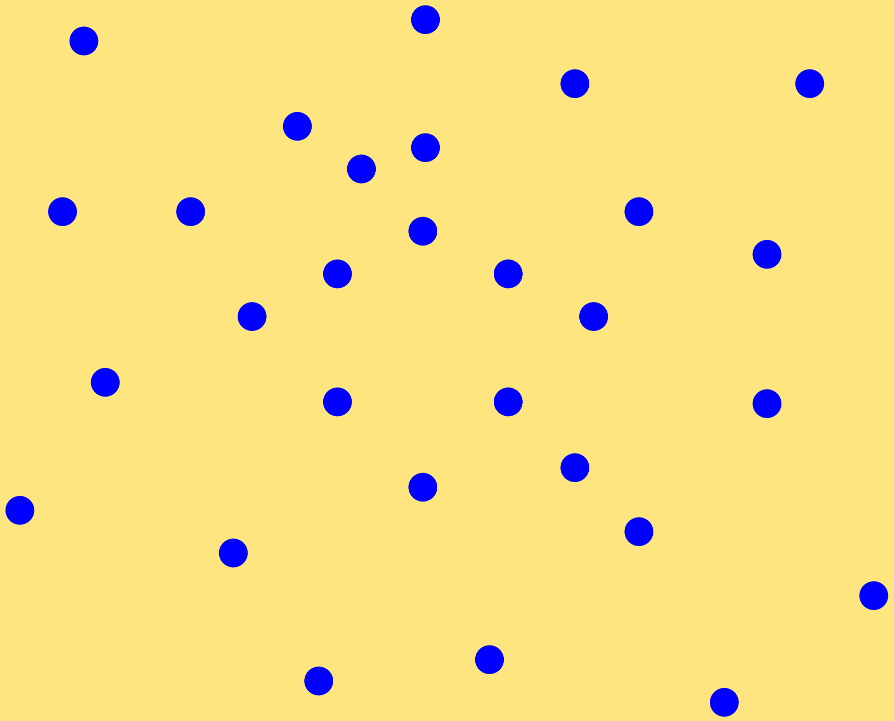
Weights $\longrightarrow$ relevance of the points

EMD for Pattern Matching

Pattern



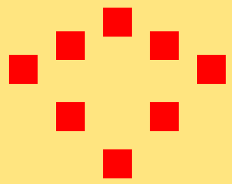
Does it appear here?



Does a given constellation appear in a photo of sky?

EMD for Pattern Matching

Pattern



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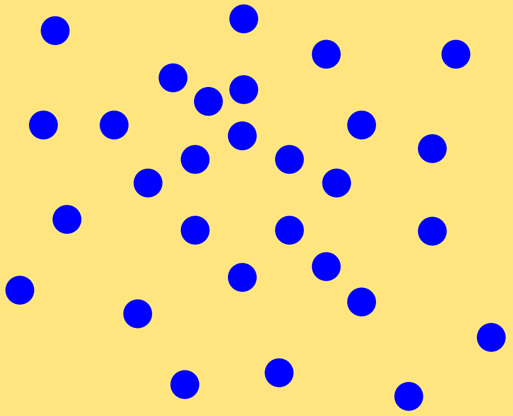
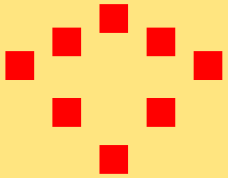


Problem: Different ”coordinate systems”.

EMD for Pattern Matching

Pattern



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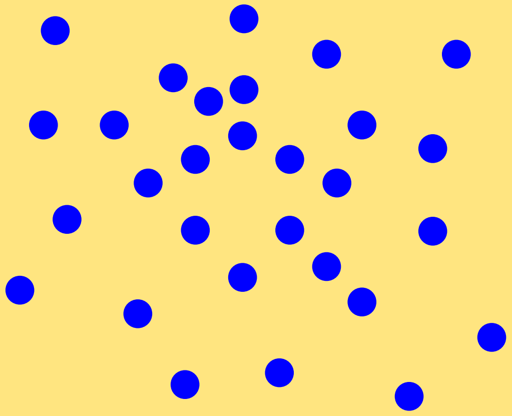

Problem: Different "coordinate systems".

Objective:

Find a rigid motion of the red set
that minimizes EMD.

The goal

A and B point sets in the plane.

EMD$(\cdot, \cdot)$ the earth moving distance between two sets.

Goal:

Find a rigid motion R^* that minimizes EMD$(R(A), B)$

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Find a global minima of the function $F : \mathcal{R} \rightarrow \mathbb{R}$

$\mathcal{R}$ set of rigid motions,

$F(R) =$ EMD$(R(A), B)$

The goal

A and B point sets in the plane.

EMD$(\cdot, \cdot)$ the earth moving distance between two sets.

Goal:

translation

Find a ~~rigid motion~~ R^* that minimizes EMD$(R(A), B)$

Find a global minima of the function $F : \mathcal{R} \rightarrow \mathbb{R}$

$\mathcal{R}$ set of ~~rigid motions,~~ translation

$F(R) =$ EMD$(R(A), B)$

EMD: the good, the bad, and the ugly

- Very good similarity measure
- Expensive to compute (without motion, min-cost flow)
- With motion, not known if computable

- We give simple $(1+\epsilon)$-approximation (without motion)
- In general, we give many simple $(1+\epsilon)$-approximations but not so efficient

- First serious results
- Still far from being applicable in practice

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  → but not so efficient

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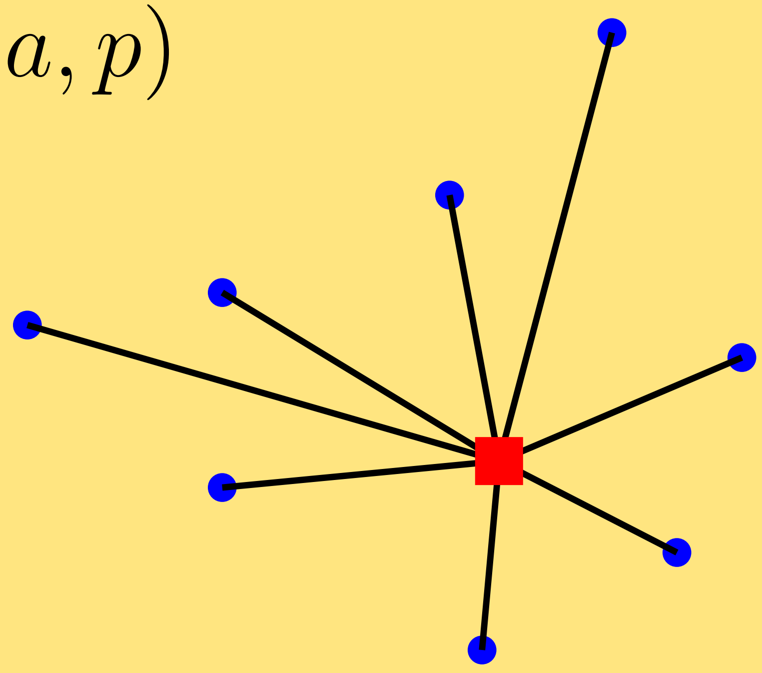
A simple case: Fermat-Weber

Given a set of points A, find $p^* \in \mathbb{R}^2$ minimizing

$$D(p) = \sum_{a \in A} d(a, p)$$



Fermat-Weber point $\equiv$ one-mean

Not known how to compute it exactly
but easy to approximate.

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Some simple deas about the results

Today's scenario:

A and B point sets with n and m points
$m \leq n$

Objective:

Find translation T_a s.t.

$$(1+\epsilon)\mathsf{EMD}(T_a(A), B) \geq \min_T \mathsf{EMD}(T(A), B)$$

Notation: $(1+\epsilon)\mathsf{EMD}(T_a) \geq \min_T \mathsf{EMD}(T)$

A couple of lemmas

$T_{a\to b}$ translation that brings $a \in A$ over $b \in B$

T^* an optimal translation

Lem:

$$\min_{a\in A, b\in B} d(T^*, T_{a\to b}) \leq \text{EMD}(T^*)$$

Pf: If each $d(T^*, T_{a\to b}) > \text{EMD}(T^*)$
$\Rightarrow$each edge in the optimal matching has length $> \text{EMD}(T^*)$
$\Rightarrow$optimal matching has length $> m\,\text{EMD}(T^*)$
$\Rightarrow$Contradiction.

A couple of lemmas

$T_{a\to b}$ translation that brings $a \in A$ over $b \in B$

T^* an optimal translation

Lem:

$$\mathsf{EMD}(T^*) \le \min_{a\in A, b\in B} \mathsf{EMD}(T_{a\to b}) \le 2\,\mathsf{EMD}(T^*)$$

Algorithm: Try each translation $T_{a\to b}$ for $(a,b) \in A \times B$; choose the best.

Obvious: This algorithm computes a 2-approximation computing nm times a EMD value.

Refine each neighbourhood

Idea: Near some $T_{a \to b}$ there is T^*.

$\Rightarrow$ Try many points close to each $T_{a \to b}$

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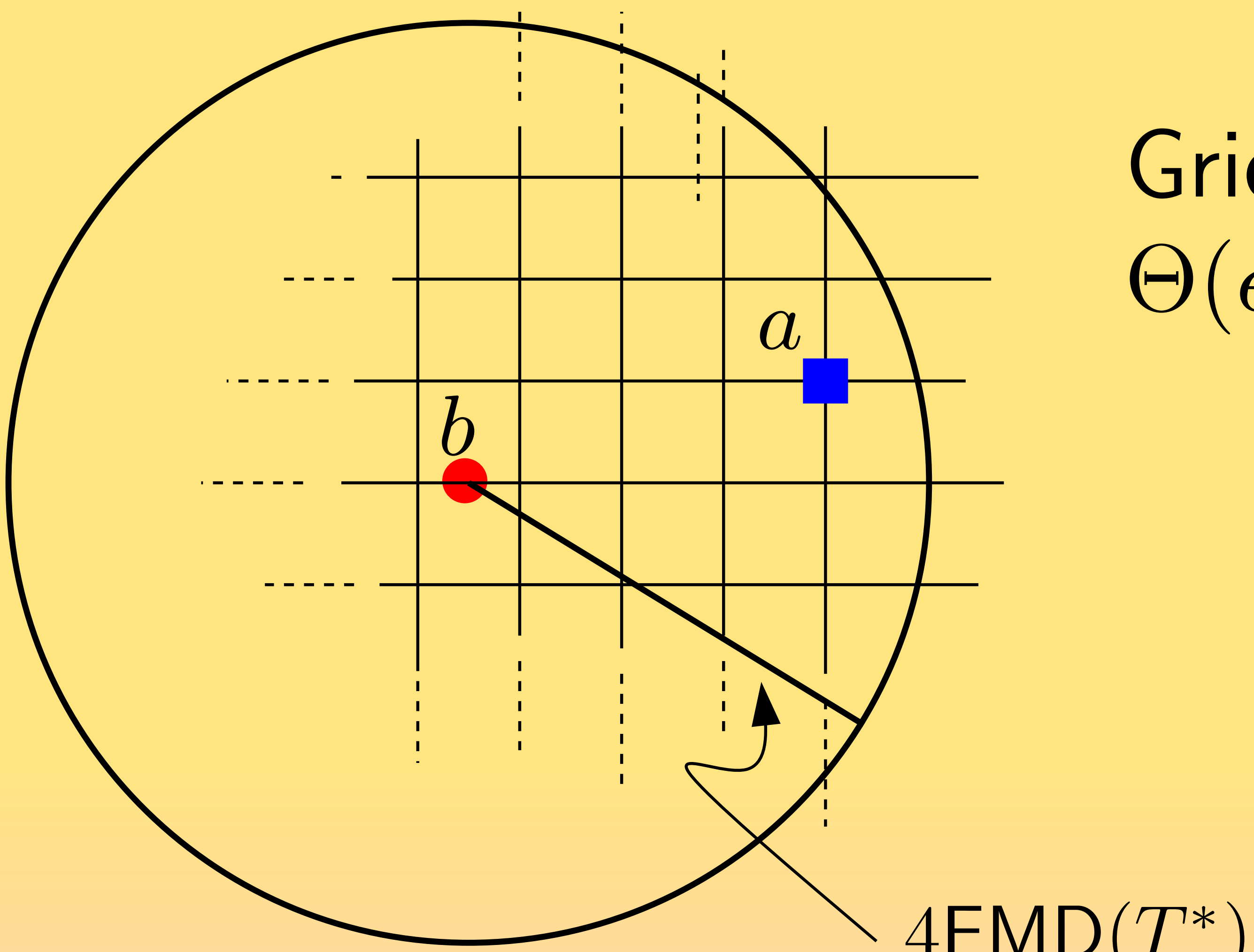

Grid size:
$\Theta(\epsilon \times \mathrm{EMD}(T^*))$

Refine each neighbourhood

Idea: Near some $T_{a \to b}$ there is T^*.

$\Rightarrow$ Try many points close to each $T_{a \to b}$

Algorithm:

for each $(a, b) \in A \times B$;

try translations in an ϵ-grid around $T_{a \to b}$

choose the best.

Not so Obvious:

This algorithm computes a $(1 + \epsilon)$-approximation computing $O(nm/\epsilon^2)$ times a EMD value.

Many pairs: probabilistic improvement

Lem: There are $m/2$ pairs $(a_1, b_1), \ldots$ such that

$$d(T^*, T_{a_i \to b_i}) \leq 2\mathsf{EMD}(T^*)$$

Algorithm:
repeat $O(n \log n)$ times
  choose random $(a, b) \in A \times B$;
  try translations in an ϵ-grid around $T_{a \to b}$
choose the best.

Not so Obvious:

With probability $1 - \frac{1}{n^c}$, this algorithm computes a $(1 + \epsilon)$-approximation
computing $O(n \log n / \epsilon^2)$ times a EMD value

Same weight: simpler and quicker

Lem: If $n = m$, then

$$d(T^*, T_{\text{align center masses}}) \leq \text{EMD}(T^*)$$

Algorithm:

try each translations in an ϵ-grid around $T_{\text{align center masses}}$

choose the best.

Obvious: This algorithm computes a $(1+\epsilon)$-approximation computing $O(1/\epsilon^2)$ times a EMD value.

Summary

- EMD: good similarity measure
- EMD for pattern matching
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THANKS!