On a L_1-test statistic of homogeneity

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Outline

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1. A L_1-test statistic for the two sample problem

2. Application to density model selection

The problem

- Two mutually independent samples

$$X_1, \ldots, X_n \quad \text{and} \quad X'_1, \ldots, X'_n$$

  distributed according to unknown probability measures μ and μ' on $\mathbb{R}^d$.
- We are interested in testing the null hypothesis that the two samples are homogeneous, that is

$$\mathcal{H}_0 : \mu = \mu'.$$

- Such tests have been extensively studied (for an overview, see Gretton, Borgwardt, Rasch, Schölkopf and Smola, 2006).

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The test statistic

- Based on a partition $\mathcal{P}_n = \{A_{n1}, \ldots, A_{nm_n}\}$ of $\mathbb{R}^d$, we let the test statistic be defined as

$$T_n = \sum_{j=1}^{m_n} |\mu_n(A_{nj}) - \mu_n'(A_{nj})|.$$

- Györfi and van der Meulen (1990) introduced a related goodness of fit test statistic L_n defined as

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Asymptotic behavior of L_n

Theorem (Devroye and Györfi, 2002)

If

$$\lim_{n\to\infty} m_n/n = 0 \quad \textit{and} \quad \lim_{n\to\infty} \max_{j=1,\ldots,m_n} \mu(A_{nj}) = 0,$$

then, for all $0 < \varepsilon < 2$*,*

$$\mathbb{P}\{L_n > \varepsilon\} = e^{-n(g_L(\varepsilon)+o(1))} \quad \textit{as } n \to \infty,$$

where

$$g_L(\varepsilon) = \inf_{0<p<1-\varepsilon/2} D(p \,\|\, p+\varepsilon/2),$$

and

$$D(\alpha \,\|\, \beta) = \alpha \ln \frac{\alpha}{\beta} + (1-\alpha) \ln \frac{1-\alpha}{1-\beta}.$$

Asymptotic behavior of T_n

Theorem

Under $\mathcal{H}_0$, for all $0 < \varepsilon < 2$,

$$\mathbb{P}\{T_n > \varepsilon\} = e^{-n(g_T(\varepsilon)+o(1))} \quad \text{as } n \to \infty,$$

where

$$g_T(\varepsilon) = (1+\varepsilon/2)\ln(1+\varepsilon/2) + (1-\varepsilon/2)\ln(1-\varepsilon/2).$$

- As $\varepsilon \downarrow 0$, $g_T(\varepsilon) \approx \varepsilon^2/4$, whereas $g_L(\varepsilon) \approx \varepsilon^2/2$.
- In contrast to $g_T(\varepsilon)$, the rate function $g_L(\varepsilon)$ is unbounded as $\varepsilon \uparrow 2$.
- Conclusion: L_n and T_n have different large deviation properties.

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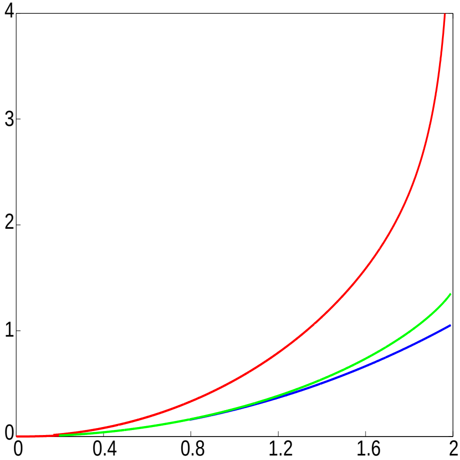
Rate functions g_L and g_T



$$g_L(\varepsilon/2) \leq g_T(\varepsilon) \leq g_L(\varepsilon)$$

Sketch of proof

- Generating function of the sequence (T_n):

$$\lambda_T(s) = \lim_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\{e^{snT_n}\}, \quad s > 0.$$

- By Scheffé's theorem for partitions:

$$T_n = \sum_{A\in\mathcal{P}_n} |\mu_n(A) - \mu'_n(A)| = 2 \max_{A\in\sigma(\mathcal{P}_n)} (\mu_n(A) - \mu'_n(A)) .$$

- Thus,

$$\begin{aligned}
\mathbb{E}\{e^{snT_n}\} &= \mathbb{E}\left\{ \max_{A\in\sigma(\mathcal{P}_n)} e^{2sn(\mu_n(A)-\mu'_n(A))} \right\} \\
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$$g_T(\varepsilon) = \max_{s>0} (s\varepsilon - \lambda_T(s)).$$

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A distribution-free strong consistent test

- This technique yields a distribution-free strong consistent test of homogeneity, which rejects the null hypothesis if T_n becomes large.
- It means that both on $\mathcal{H}_0$ and on its complement the test makes a.s. no error after a random sample size.
- In other words, we have

$$\mathbb{P}_0\{\text{rejecting } \mathcal{H}_0 \text{ for only finitely many } n\} = 1$$

and

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A strong consistent test

Corollary

Consider the test which rejects $\mathcal{H}_0$ when

$$T_n > c_1\sqrt{\frac{m_n}{n}},$$

where $c_1 > 2\sqrt{\ln 2} \approx 1.6651$. *Assume that*

$$\lim_{n\to\infty} m_n/n = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{m_n}{\ln n} = \infty.$$

Then, under $\mathcal{H}_0$, after a random sample size the test makes a.s. no error. Moreover, if $\mu \neq \mu'$, and for any sphere S centered at the origin

$$\lim_{n\to\infty} \max_{A_{nj}\cap S\neq 0} diam(A_{nj}) = 0,$$

then after a random sample size the test makes a.s. no error.

Asymptotic normality

- Beirlant, Györfi and Lugosi (1994) proved that

$$\sqrt{n}\left(L_n - \mathbb{E}\{L_n\}\right)/\sigma \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

  where $\sigma^2 = 1 - 2/\pi$.

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then, under $\mathcal{H}_0$*, with a centering sequence* (C_n)*,*

$$\sqrt{n}\,(T_n - C_n)\,/\sigma \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

where $\sigma^2 = 2(1 - 2/\pi)$*.*

Sketch of proof

- Difficulty: T_n is a sum of dependent random variables.
- To overcome this problem, we use a 'Poissonization' argument.
- Denote by N_n and N'_n two independent Poisson (n) random variables independent of $(X_i)_{i\geq 1}$ and $(X'_i)_{i\geq 1}$.
- The Poissonized version $\tilde{T}_n$ of T_n is then defined by

$$\tilde{T}_n = \sum_{j=1}^{m_n} |\mu_{N_n}(A_{nj}) - \mu'_{N'_n}(A_{nj})|,$$

where, for any Borel subset A,

$$\mu_{N_n}(A) = \frac{\#\{i : X_i \in A, i = 1, \ldots, N_n\}}{n},$$

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  and

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  one shows that $\mathbf{Y}_n$ and $\mathbf{Y}'_n$ are independent vectors of independent random variables with

  $$\left(n\mu_{N_n}(A_{nj})\right) \stackrel{\mathcal{D}}{=} \left(n\mu'_{N'_n}(A_{nj})\right) \stackrel{\mathcal{D}}{=} \text{Poisson}\left(n\mu(A_{nj})\right).$$

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Proposition

Let g_{nj} $(j = 1, \ldots, m_n)$ be real measurable functions, with

$$\mathbb{E}\left\{g_{nj}\left(\mu_{N_n}(A_{nj}) - \mu'_{N'_n}(A_{nj})\right)\right\} = 0,$$

and let

$$M_n = \sum_{j=1}^{m_n} g_{nj}\left(\mu_{N_n}(A_{nj}) - \mu'_{N'_n}(A_{nj})\right).$$

Assume that

$$\left(M_n, \frac{N_n - n}{\sqrt{n}}, \frac{N'_n - n}{\sqrt{n}}\right) \xrightarrow{\mathcal{D}} \mathcal{N}_3(0, 0, 0, \sigma^2, 1, 1),$$

as $n \to \infty$, where σ is a positive constant. Then

$$\frac{1}{\sigma}\sum_{j=1}^{m_n} g_{nj}\left(\mu_n(A_{nj}) - \mu'_n(A_{nj})\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

A consistent test

Corollary

Put $\alpha \in (0,1)$, $C^* = 0.7655$, and consider the test which rejects $\mathcal{H}_0$ when

$$T_n > c_2\sqrt{\frac{m_n}{n}} + C^*\frac{m_n}{n} + \frac{\sigma}{\sqrt{n}}\Phi^{-1}(1-\alpha),$$

where $c_2 = 2/\sqrt{\pi} \approx 1.1284$. Then the test has asymptotic significance level α. Moreover, under the additional condition

$$\lim_{n\to\infty} \max_{A_{nj}\cap S\neq 0} \mathit{diam}(A_{nj}) = 0,$$

the test is consistent.

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The problem

- We wish to estimate a density f on $\mathbb{R}^d$ that belongs to a parametric family, $\mathcal{F}_k$, where k is unknown, but $\mathcal{F}_k \subset \mathcal{F}_{k+1}$ for all k.

$$\mathcal{F} = \bigcup_{k \geq 1} \mathcal{F}_k.$$

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Objective

- We wish to pick a density estimate $\hat{f}_{K_n}$ in $\mathcal{F}$ with
  (i) $K_n \to k^*$ almost surely
  (ii) and
$$\mathbb{E}\left\{\int |\hat{f}_{K_n} - f|\right\} = \mathrm{O}\left(\frac{1}{\sqrt{n}}\right).$$
- K_n is obtained by minimizing the L_1 error between candidate models and the empirical measure.
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Examples I

Mixture classes. Consider first the classes $\mathcal{F}_k$ of all mixtures of k normal densities over $\mathbb{R}^d$,

$$f_k(x) = \sum_{i=1}^{k} \frac{p_i}{\sqrt{(2\pi)^d \det(\Sigma_i)}} e^{-\frac{1}{2}(x-m_i)^T \Sigma_i^{-1}(x-m_i)}.$$

- $\longrightarrow$ Bayesian literature: Hurn, Justel and Robert (2003).
- $\longrightarrow$ Statistical learning literature: Figueiredo and Jain (2002).
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$$f_k(x) = c\alpha(\theta)\beta(x)e^{\sum_{i=1}^{k}\pi_i(\theta)\psi_i(x)}.$$

- Examples of exponential families include classes of Gaussian, gamma, beta, Rayleigh, and Maxwell densities.
- Other models are feasible: series estimates, neural network estimates, wavelets...
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A closure condition

- Let $\mathcal{D}$ be the class of all density functions on $\mathbb{R}^d$ and $\hat{\mathcal{D}}$ the set of Fourier transforms $\hat{g}$.

Assumption

The set $\hat{\mathcal{F}}_k$ is closed in $\hat{\mathcal{D}}$.

- By Paul Lévy's theorem, this is equivalent to require that for any sequence (g_n) in $\mathcal{F}_k$ satisfying

$$\lim_{n\to\infty} \int g_n(x)\varphi(x)\,\mathrm{d}x = \int g(x)\varphi(x)\,\mathrm{d}x$$

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Complexity estimation

- Split the sample into two subsamples:

$$\{X_1, \ldots, X_n\} \quad \text{and} \quad \{X'_1, \ldots, X'_n\} = \{X_{n+1}, \ldots, X_{2n}\}.$$

- Let $\mathcal{P}_n = \{A_{nj} : j \geq 1\}$ be a cubic partition of $\mathbb{R}^d$ with volume h_n^d.
- Introduce the statistic

$$d_{n,k} = \inf_{g \in \mathcal{F}_k} \sum_{A \in \mathcal{P}_n} \left| \int_A g - \mu_{2n}(A) \right|.$$

- Let the threshold be

$$T_n = \sum_{A \in \mathcal{P}_n} |\mu_n(A) - \mu'_n(A)|.$$

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- Our estimate of k^*:

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Theorem

Choose $h_n = n^{-\delta}$ with $0 < \delta < 1/d$. Then there exists a positive constant κ, depending on f, such that

$$\mathbb{P}\{K_n \neq k^*\} \leq \exp\left(-\kappa n^{d\delta}\right),$$

and consequently, almost surely,

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Fast density estimate

- Fix $k \geq 1$ and introduce the class of sets

$$\mathcal{A}_k = \big\{\{x : g_1(x) > g_2(x)\} : g_1, g_2 \in \mathcal{F}_k\big\}$$

and the goodness criterion for a density $g \in \mathcal{F}_k$:

$$\Delta_k(g) = \sup_{A \in \mathcal{A}_k} \left| \int_A g - \mu_{2n}(A) \right|.$$

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- For the elected minimum distance estimate $\hat{f}_k$, we have [Devroye and Lugosi (2001)]

$$\int |\hat{f}_k - f| \leq 3 \inf_{g \in \mathcal{F}_k} \int |g - f| + 4\Delta_k(f) + \frac{3}{2n}.$$

- The minimum distance estimate $\hat{f}_{K_n}$ is a natural candidate for the estimation of f.
- We deduce that

$$\mathbb{E}\left\{ \int |\hat{f}_{K_n} - f| \right\} \leq 4\mathbb{E}\{\Delta_{k^*}(f)\} + \frac{3}{2n} + 2\exp\left(-\kappa n^{d\delta}\right),$$

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- If $\mathcal{A}_{k^*}$ has Vapnik-Chervonenkis dimension V_{k^*}, then

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Discussion

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  - $\rightarrow$ $V_k \leq k + 1$ for the exponential families.
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On the closure condition

- For any set of parameters $\Theta \subset \mathbb{R}^s$ and any density $\psi(.,\theta)$ defined on $\mathbb{R}^d \times \Theta$, let the collection $\mathcal{C}_\psi$ be

$$\mathcal{C}_\psi = \{\psi(.,\theta) : \theta \in \Theta\}.$$

Proposition

Assume that

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Suppose that $\hat{\mathcal{C}}_\psi$ is closed in $\hat{\mathcal{D}}$, and consider the k-th mixture class associated with ψ defined by

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