CONFIGURATIONS FROM GRAPH-THEORETICAL VIEWPOINT

Notes for a book

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CONTENTS

TWO COURSES

- ALGEBRAIC AND TOPOLOGICAL GRAPH THEORY
  - CHAPTER 0 - Introduction
  - CHAPTER 1 - Graphs
  - CHAPTER 2 - Groups and Symmetry
  - CHAPTER 3 - Surfaces and Maps
- COMBINATORIAL ANF GEOMETRIC CONFIGURATIONS
  - CHAPTER 4 - Incidence Geometries and Combinatorial Configurations
  - CHAPTER 5 - Geometric Configurations

More Info

- http://ucilnica.fmf.uni-lj.si/course/view.php?id=124

INTRODUCTION TO ALGEBRAIC AND TOPOLOGICAL GRAPH THEORY

- First semester
- Responsible: Primož Potočnik
- Features:
    - Exercises and Homework: Alen Orbanić
    - Guest Lecturers
    - Advanced use of e-lecture room
- Objectives:
    - Broad introduction to the subject with selected deep topics
    - Give background for the second course on configurations
    - Introduce students to research and writing of scientific papers

Exercises and Homework

- There is a special period assigned for exercises and homework.
- Each week there will be homework assigned in class and on the internet.
- The homework is due in one week and may be handed in over the internet.

Chapter 0

- 1. Motivation
- 2. Permutations

Chapter 1

1. Families of simple graphs
2. Metric space.
3. Valence, Grith and Cages
4. Isomorhpism, Matrices and Graph Invariants
5. Connectivity in Graphs
6. Subgraphs
7. Basic Operations of graphs
8. Advanced Operations on Graphs
9. Variations of Graphs
10. Graph Products

1. Factors and Factorizations
2. Planar Graphs
3. Graphs from Polyhedra
4. Metric Space - Revisited
5. Representations of Graphs
6. Edge-Colorings and Snarks
7. Vertex Colorings

Chapter 2

1. Groups and subgroups
2. Group actions
3. Symmetry groups
4. Covering graphs
5. Symmetry in graphs
6. Symmetry in metric spaces
7. Representation of graphs

Chapter 3

1. Motivation
2. Polytopes and polytopal complexes
3. Surfaces
4. Classification of surfaces
5. Connected Sum and Fundamental Polygon
6. Pseudosurfaces

1. Torus in space
2. Maps
3. Flag Systems
4. Operations on maps
5. Petrie dual
6. Symmetry of maps
7. Closed curves on a surface

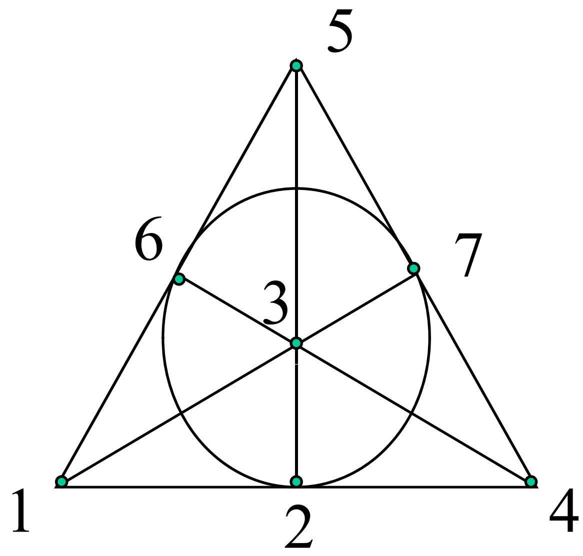
INTRODUCTION

CHAPTER 0

CONTENTS

1. Motivation

Motivation



a	b	c	d	e	f	g
1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

Example 1: Fano configuration (smallest projective plane).

lines	a	b	c	d	e	f	g
points	1	2	3	4	5	6	7

Incidence structure

- **An incidence structure** $\mathcal{C}$ is a triple
  - $\mathcal{C} = (P, \mathcal{L}, I)$ where
    - P is the set of **points,**
    - $\mathcal{L}$ is the set of **blocks or lines**
  - $I \subseteq P \times \mathcal{L}$ is an **incidence relation**.
  - Elements from I are called **flags**.
- The bipartite incidence graph $\Gamma(\mathcal{C})$ with black vertices P, white vertices $\mathcal{L}$ and edges I is known as the **Levi graph** of the structure $\mathcal{C}$.

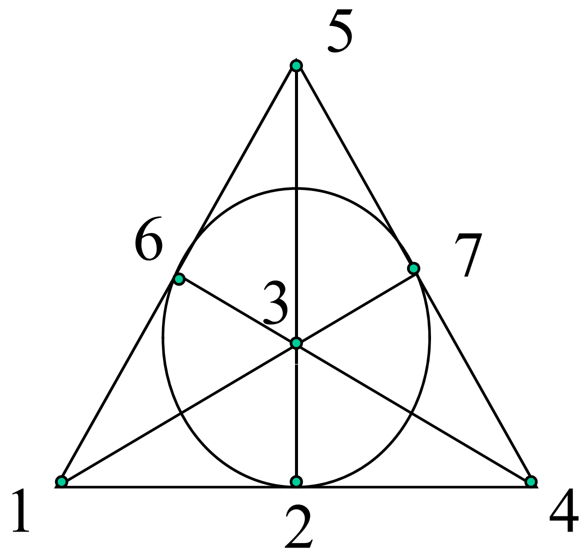
(Combinatorial) Configuration

- A (v_r, b_k) configuration is an incidence structure $\mathcal{C} = (P, \mathcal{L}, I)$ of points and lines, such that
  - $v = |P|$
  - $b = |\mathcal{L}|$
  - Each point lies on r lines.
  - Each line contains k points.
  - Two lines intersect in at most one point.
- **Warning**: Levi graph is semiregular of girth ≥ 6

Symmetric configurations

- A (v_r,b_k) configuration is **symmetric**, if
  - $v = b$ (this is equivalent to $r = k$).
  - A (v_k,v_k) configuration is usually denoted by (v_k) configuration.

Motivation



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Example 1: Fano configuration (smallest projective plane).

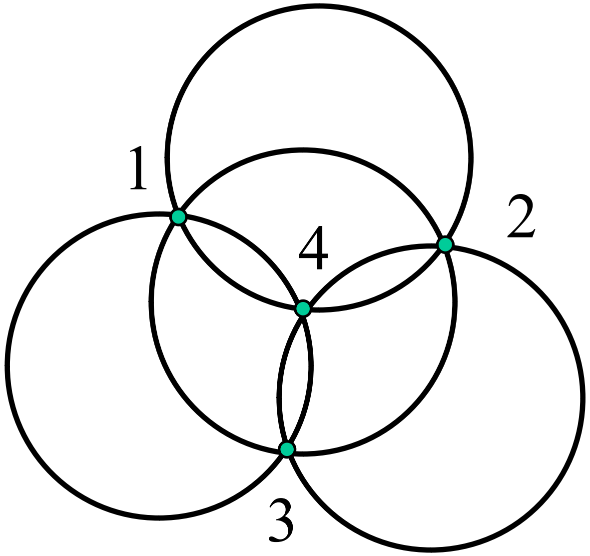
lines	a	b	c	d	e	f	g
points	1	2	3	4	5	6	7

Configuration Table

- Configuration table:
- Entries – points
- Columns - lines

a	b	c	d	e	f	g
1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

Another Example



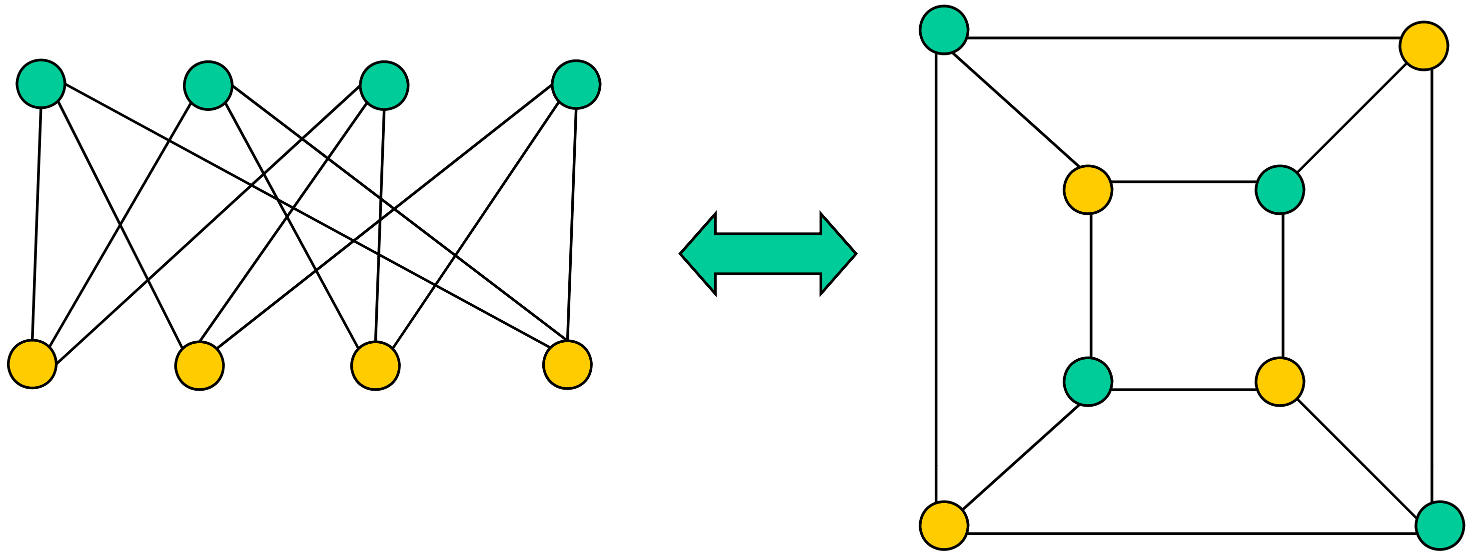
- The figure on the left has 4 points 1,2,3,4 and four circles a,b,c,d. A combinatorial description is given by the following configuration table.

a	b	c	d
1	1	1	2
2	3	2	3
4	4	3	4

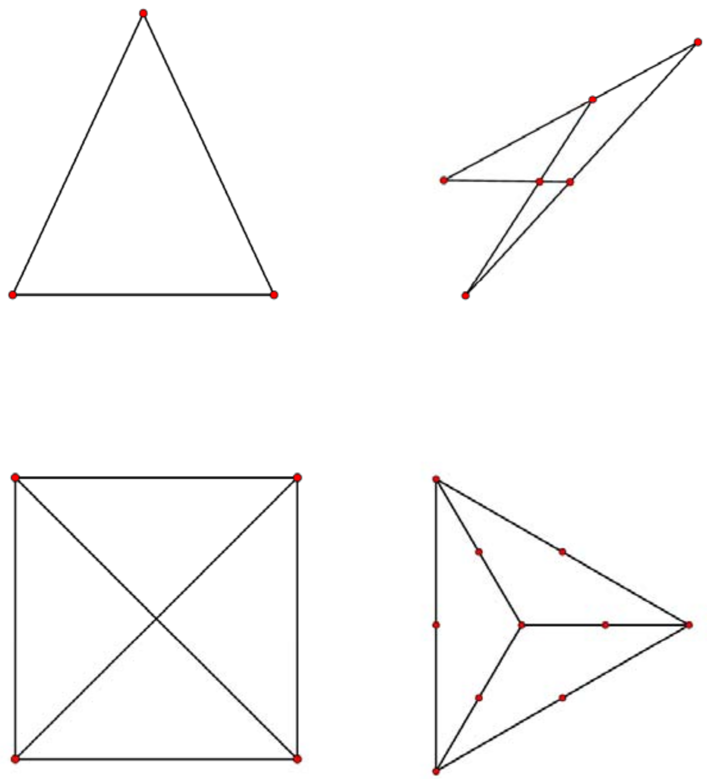
Another Example - Continuation

a	b	c	d
1	1	1	2
2	3	2	3
4	4	3	4

- The configuration table admits a "graphical" description.

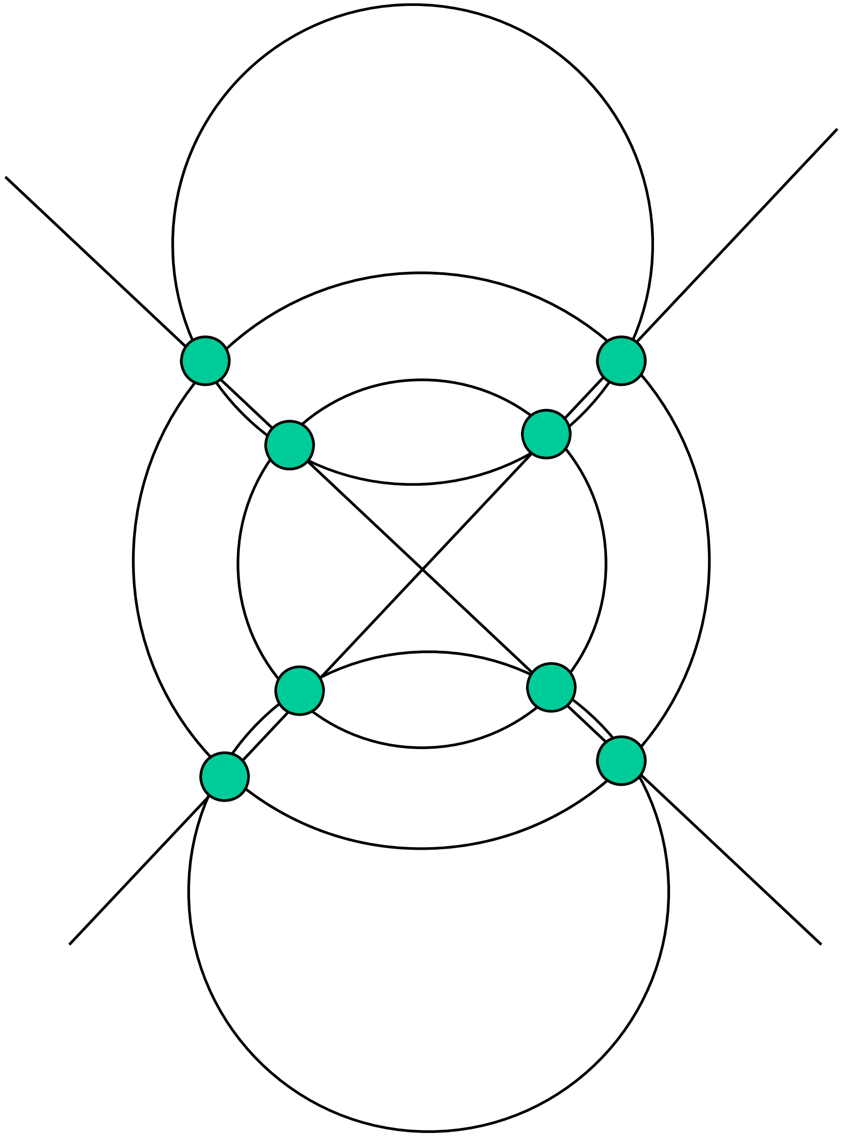


Small Configurations



- **a triangle** is the only (3_2) configuration.
- **the Pasch configuration** $(6_2,4_3)$ and its dual, the **complete quadrangle** $(4_3,6_2)$, share the same Levi graph $S(K_4)$.

Miquel "Configuration"



- This configuration constists of 8 points and 6 circles (see figure on the left.) It is called the Miquel configuration. [Note that two lines are considered as "degenerate circles"]
- Configuration results from the Miquel theorem.

Examples

- 1. Each graph G = (V,E) is an incidence structure: P = V, L = E, (x,e) 2 I if and only if x is an endvertex of e.
- 2. Any family of sets F µ P(X) is an incidence structure. P = X, L = F, I = 2.
- 3. A line arrangement L = $\{l_1, l_2, ..., l_n\}$ consisting of a finite number of n distinct lines in the Euclidean plane E^2 defines an incidence structure. Let V denote the set of points from E^2 that are contained in at least two lines from L. Then: P = V, L = L and I is the point-line incidence in E^2.

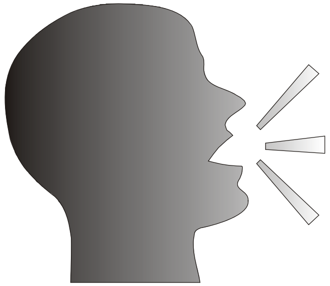
Media Coverage of Presidential Elections

- In Slovenia in 1997 there were 8 presidential candidates: Bernik, Cerar, Kovač, Kučan, Miklavčič, Peršak, Podobnik, Polišak

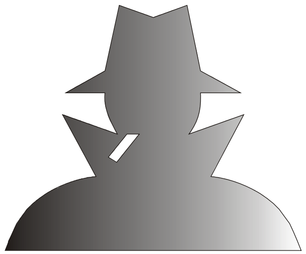
- Debates on TV Slovenija: 8 evenings, each evening 3 candidates

Presidential Elections – Biased Media Coverage

One eveninig in the TV studio



Seven days later ...



Question

- Can we group the 8 candidates into 8 triples in such a way that no 2 of them will meet more than once?

Trivalent Combinatorial Configurations

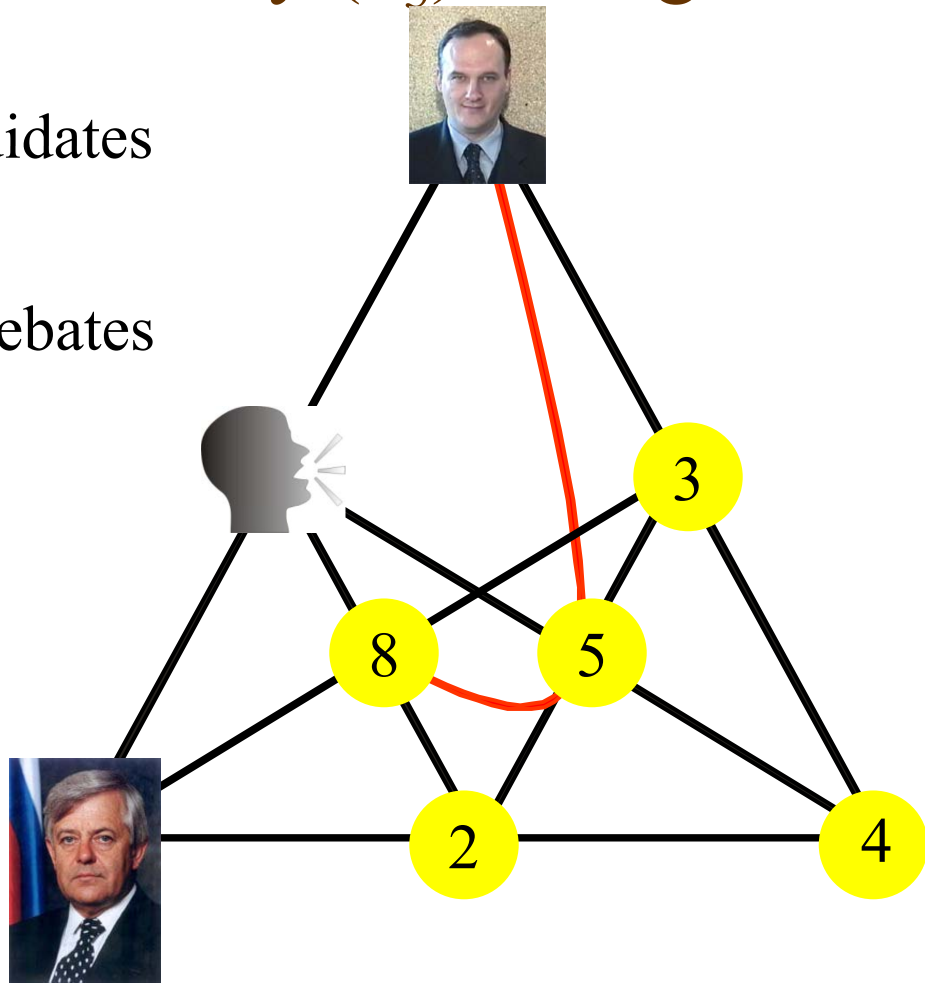
- We need a mathematical structure with the following properties:
- It has two types of objects:
    - Points (presidential candidates)
    - Lines (TV-debates)
    - Each point lies on exactly three lines
    - Each line passes through exactly three points.
    - Two points lie on at most one common line.
- Such a structure is called a **trivalent combinatorial configuration**. If v is the number of points (and lines) then it is denoted as a (v_3) configuration.

Möbius - Kantor configuration

The only (8_3) configuration

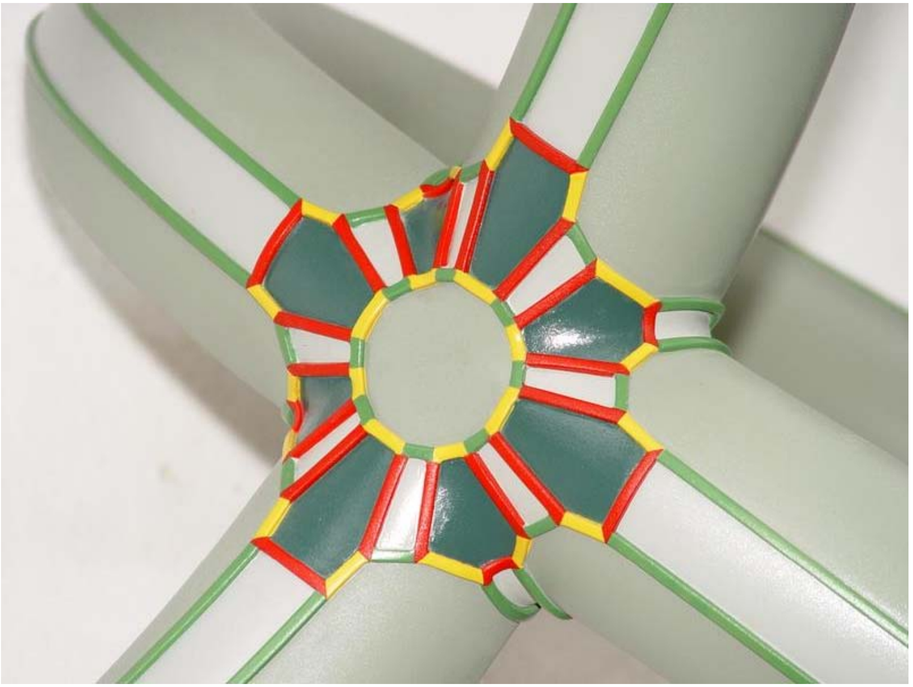
Points: 8 candidates

Lines: 8 TV-debates



124
235
346
457
568
671
782
813

A Surprising connection



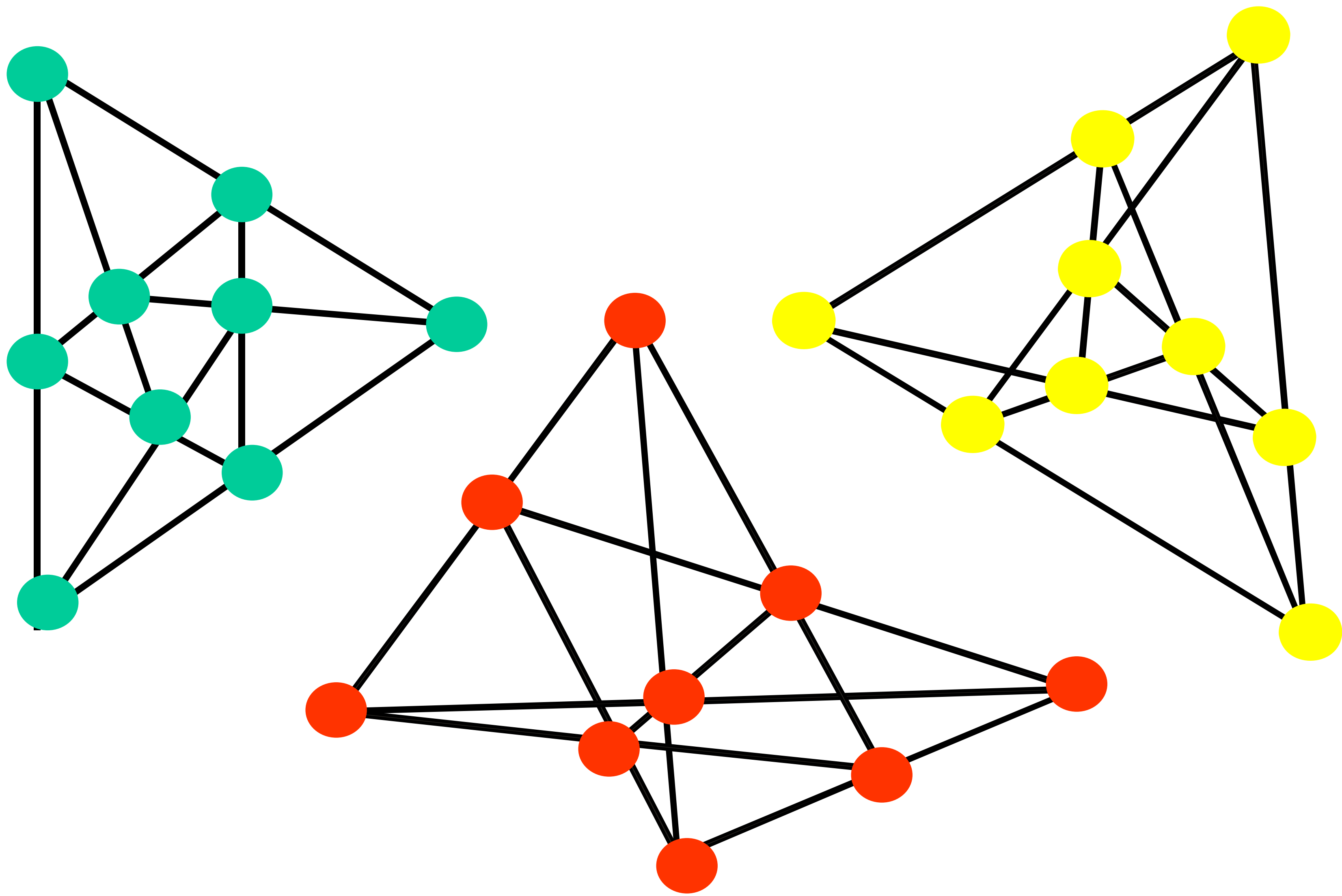
- There is deep connection between scheduling the 8 candidates and the sculptrue on the left.

Changed condition

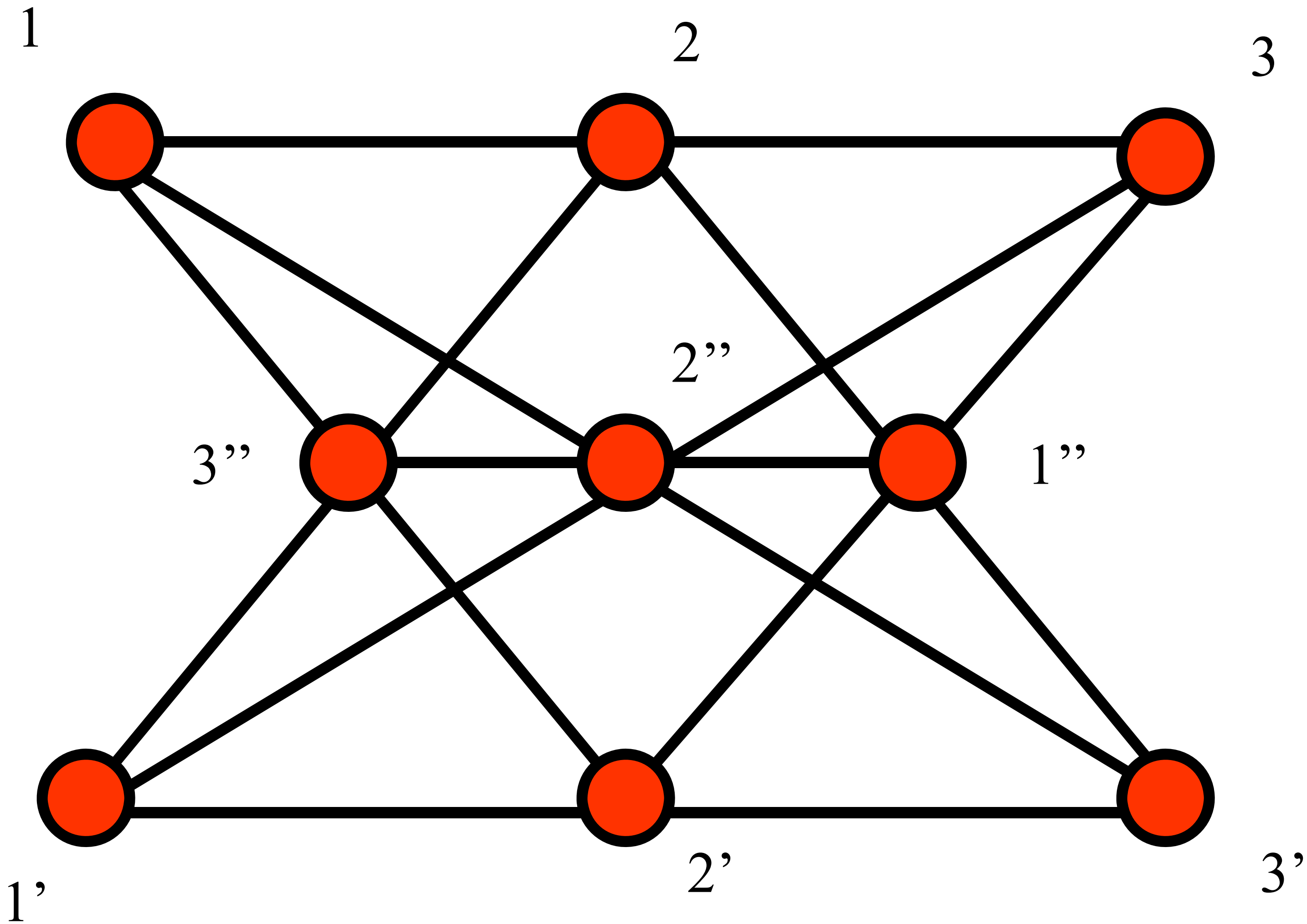
- Assume that there are 9 presidentinal candidates.

- Is it possible to group the nine candidates into 9 triples in such a way, that no two of them meet more than once?

All (9_3) configurations



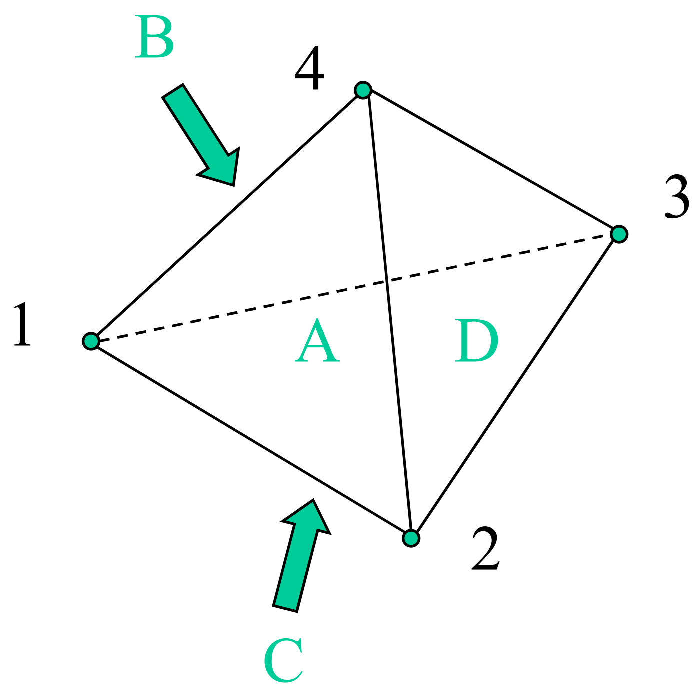
Pappus Configuration



Exercises 01

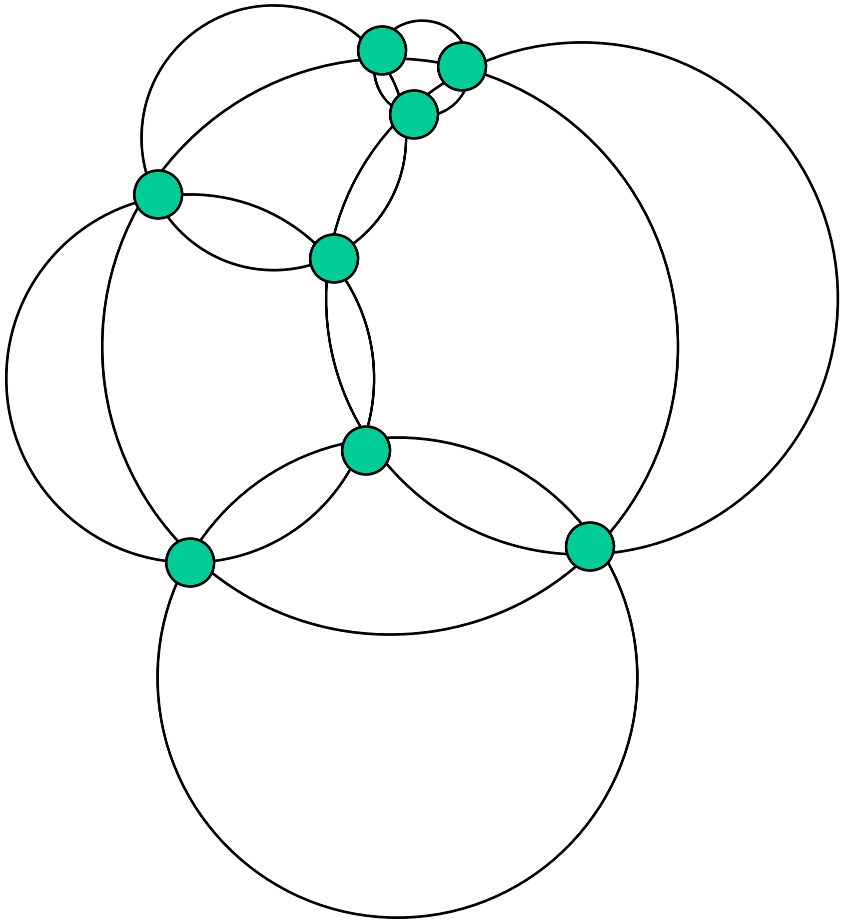
- N1: Determine all lines passing through the point 3 of the Fano plane.
- N2: How many lines pass through 4 and 6?
- N3: How many lines pass through 1, 2, 3?
- N4*: Is it possible to draw the Fano plane with straight lines?
- N5*: Describe various properties of the Fano plane. For example: Each number from 1 to 7 appears exactly three times in the table.

Another Example – Exercises 02



- N6: Determine the configuration table for the vertices and faces of the tetrahedron.
- N7: Determine the configuration table for the vertices and edges of the tetrahedron.
- N8: Explore the relationship between these structures..

Miquel Configuration – Exercises 03



- N9: Show that the "configuration" on the left is not combinatorially equivalent to the Miquel configuration.
- N10: How many points and curves does the "configuration" on the left have?

Exercises 04

- N11. Draw the Levi graph of the incidence structure defined by the complete bipartite graph $K_{3,3}$.
- N12. Draw the Levi graph of the incidence structure defined by the powerset $\mathsf{P}(\{a,b,c\})$.
- N13. Determine the Levi graph of the incidence structure, defined by an arrangemnet of three lines forming a triangle in E^2.

Exercises 05

- N14: How many points and lines does the Pappus configuration have?
- N15: How many lines pass through the same point of the Pappus configuration?
- N16: What is the maximum number of lines common to the same pair of points?
- N17: Write down the configuration table for the Pappus configuration.
- N18*: Plant 9 trees in 10 lines, in such a way that each line contains exactly 3 trees!

Exercises 06

- N19. Show that the configuration table of N17 is equivalent to the following table:

1	2	3	4	5	6	7	8	9
8	7	2	1	6	9	3	4	5
6	1	8	9	7	3	4	5	2

Homework 01

- H1. Define an isomorphism of incidence structures.
- H2. Two configuration tables are equivalent if they represent isomorphic incidence structures. Write down conditions that check whether two configuration tables are equivalent.
- H3. List all properties of a configuration table of a (v_r) configuration.
- **H4. Write down a configuration table of the Miquel "configuration."**
- **H5. Write down a configuration table for a "configuration" defined by the set of vertices and faces of the cube.**
- **H6. Show that the tables from H4 and H5 are equivalent.**

2. Permutations

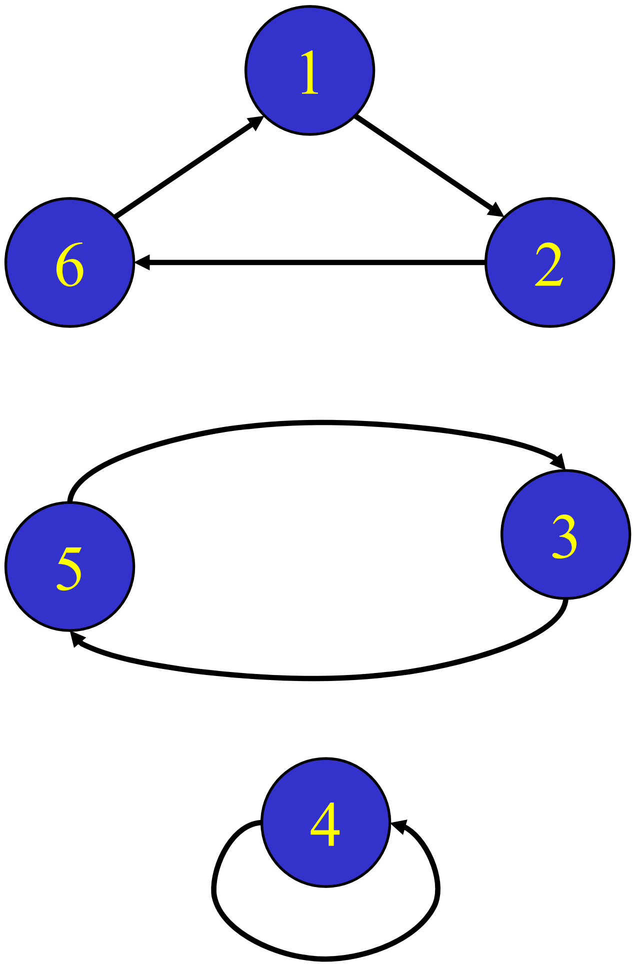
Permutations

- As we know a permutation π is a bijective mapping of a set A onto itself: π: A $\rightarrow$ A. Permutations may be multiplied and form the symmetric group Sym(A) = S_A, which has n! elements, where n = |A|.

Permutation as a product of disjoint cycles

- A permutation can be written as a product of disjoint cycles in a uniqe way.

Example



- Example:
  - $\pi(1) = 2$, $\pi(2) = 6$,
  - $\pi(3) = 5$, $\pi(4) = 4$,
  - $\pi(5) = 3$, $\pi(6) = 1$.
- Permutation π can be written as:
- $\pi = [2,6,5,4,3,1]$
- $\pi = (1\ 2\ 6)(3\ 5)(4)\ 2\ S_6$

Positional Notation

$\pi = [2,6,5,4,3,1]$

Each row of the configuration table of the Pappus configuration is a permutation.

1	2	3	4	5	6	7	8	9
8	7	2	1	6	9	3	4	5
6	1	8	9	7	3	4	5	2

Example

- Each row of the configuration table of the Pappus configuration can be viwed as permutation (written in positional notation.)
- $\alpha_1 = (1)(2)(3)(4)(5)(6)(7)(8)(9)$
- $\alpha_2 = (184)(273)(569)$
- $\alpha_3=(16385749)$

Cyclic Permutation

- A permutation composed of a single cycle is called **cyclic permutation**.
- For example, α_3 is cyclic.

Polycyclic Permutation

- A permutation whose cycles are of equal length is called **semi-regular** or **polycyclic**.
- Hence α_1, α_2 and α_3 are polycyclic.

Identity Permutation

- A polycyclic permutation with cycles of length 1 is called **identity permutation**.

Fix(π)

- Let $\mathrm{Fix}(\pi) = \{x\ 2\ A|\ \pi(x) = x\}$ denote the set of fixed points of permutation π.
- $\mathrm{fix}(\pi) = |\mathrm{Fix}(\pi)|$.
- a fixed-point free premutation is called a **derangment**.

Order of π.

- The **order** of a permutation π is the least number of times one has to compose π with itself to obtain the identity.
- The order of a cyclic permutation is the cycle length.
- The order of a permutation consisting of disjoint cycles is the least common multiple of the cycle lengths.

Involutions

- Permutation of order 2 is called an **involution**.
- Later we will be interseted in fixed-point free involutions.

Homework

- H1. Determine the number of polycylic configurations of degree n and order k.
- H2. Show that an ivolution is fixed-point free if and only if it is a polycyclic permutation composed of cycles of length 2.
- H3. Determine the number of involutions in S_n.