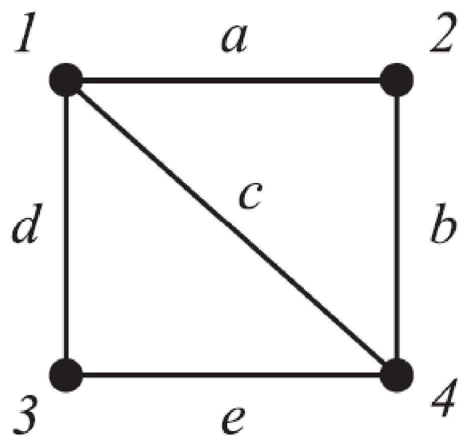
4. Isomorphism, Matrices and Graph Invariants

Incidence Matrix M(G).

To $\mathrm{G} = (V, E)$ we associate a rectangular **incidence matrix** $M = M(G)$ with $|V|$ rows and $|E|$ columns:

$$M_{v,e} = \begin{cases} 1 & \ldots \quad v \text{ is the endpoint of } e, \\ 0 & \ldots \quad \text{otherwise.} \end{cases}$$



$G = (V, E)$
$V(G) = \{1, 2, 3, 4\}$
$E(G) = \{a, b, c, d, e\}$

	a	b	c	d	e
1	1	0	1	1	0
2	1	1	0	0	0
3	0	0	0	1	1
4	0	1	1	0	1

Handshaking Lemma

Lemma (Handshaking lemma)

In each graph $G = (V, E)$:

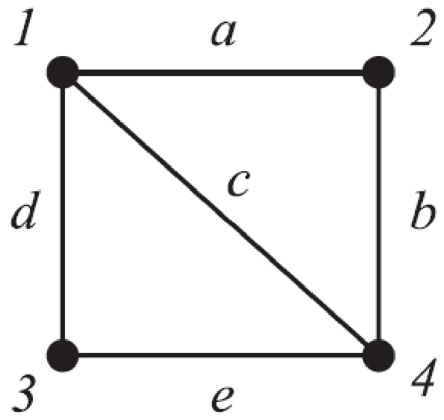
$$2|E(G)| = \sum_{v \in V(G)} \mathrm{val}(v)$$

The proof uses the so-called bookkeepers rule in the incidence matrix of graph G.

Adjacency matrix

To each graph $G = (V, E)$ with $V = \{1, 2, 3, ..., n\}$ we can associate the **adjacency matrix** $A = A(G)$ as follows:

$$A_{i,j} = \begin{cases} 1 & \ldots \quad i \sim j, \\ 0 & \ldots \quad \text{otherwise.} \end{cases}$$



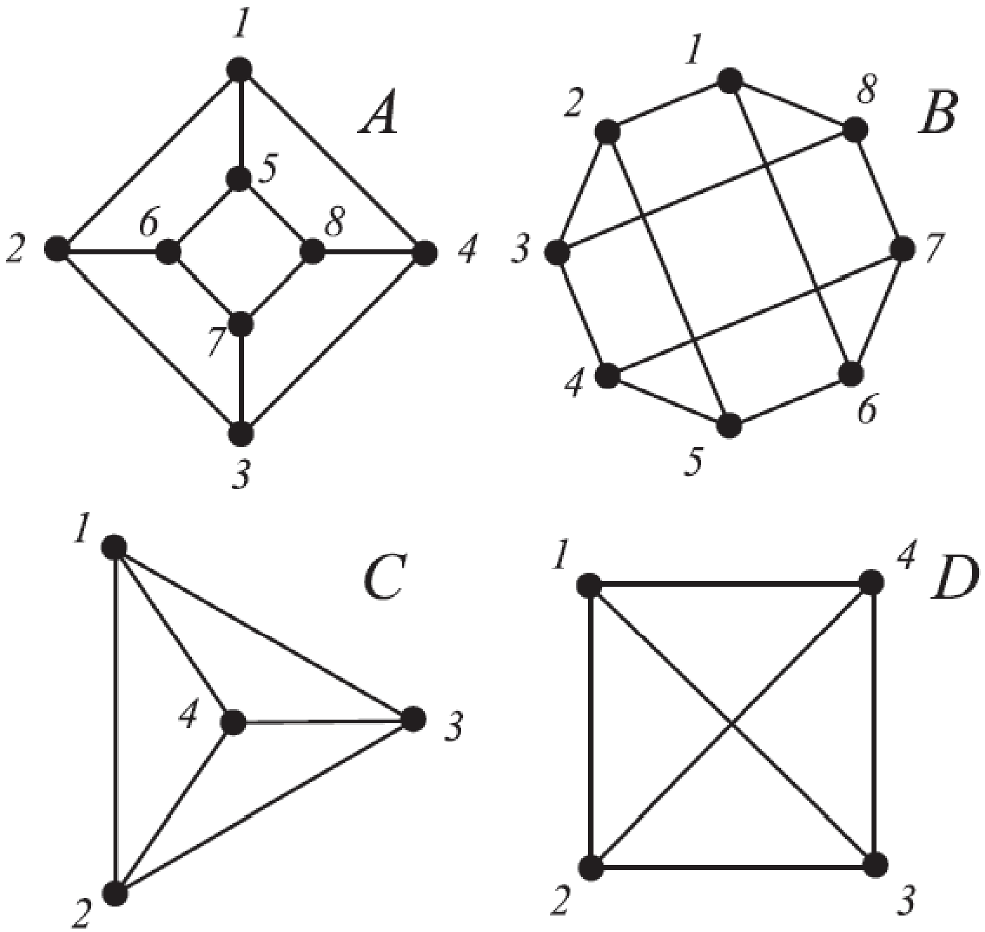
$G = (V, E)$
$V(G) = \{1, 2, 3, 4\}$
$E(G) = \{a, b, c, d, e\}$

	1	2	3	4
1	0	1	1	1
2	1	0	0	1
3	1	0	0	1
4	1	1	1	0

Isomorphisms and Graph Invariants

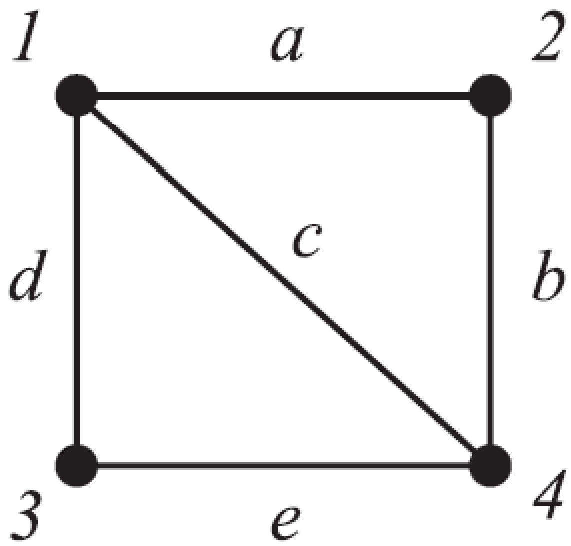
- An **isomorphism** $\sigma(G) = H$ is a bijective mapping $\sigma : V(G) \mapsto V(H)$ that preserves adjacency: $u \sim v$ if and only if $\sigma(u) \sim \sigma(v)$.
- A **graph invariant** is a property (usually a number), that is preserved under an isomorphism. Examples:
  - $|V(G)|$ - number of vertices
  - $|E(G)|$ - number of edges
  - $\delta(G)$ - minimal valence
  - $\Delta(G)$ - maximal valence

Isomorphism - Exercises



- N1 Determine an isomorphism between graphs A and B.
- N2 Determine an isomorphism between graphs C and D.

Invariants - Example



Graph invariants:

- $|V(G)| = 4$
- $|E(G)| = 5$
- $\delta(G) = 2$
- $\Delta(G) = 3$

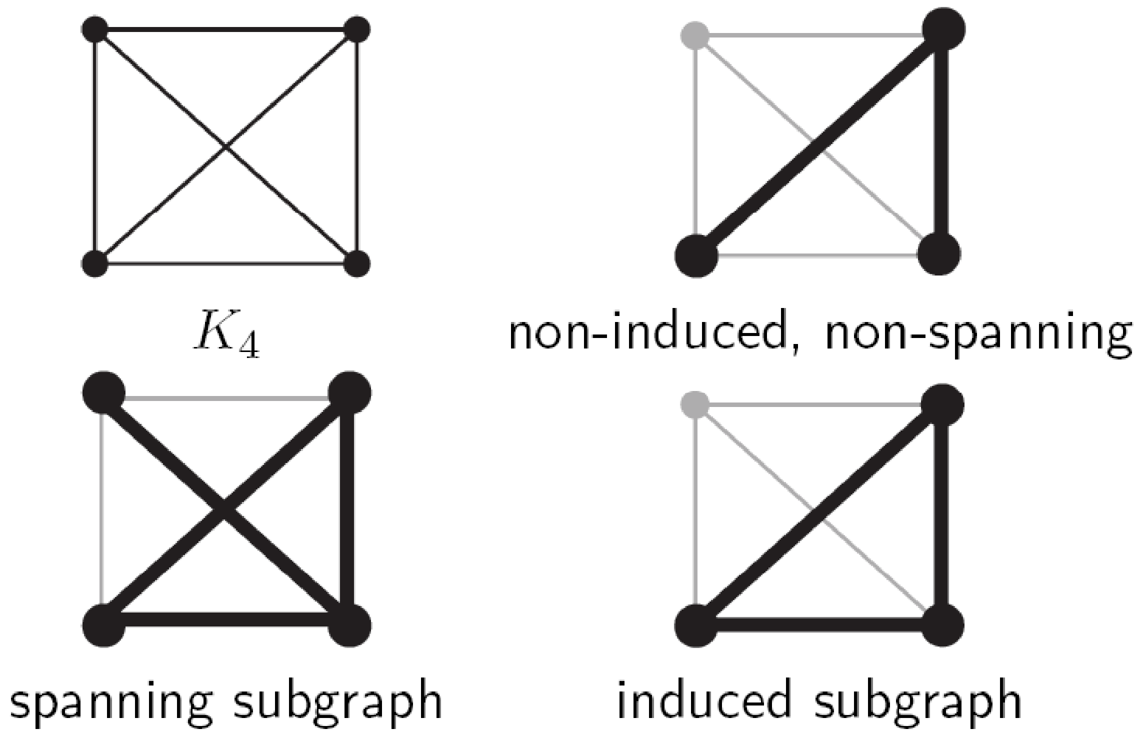
Incidence and adjacency matrices are not graph invariants - we can renumber vertices and edges.

5. Subgraphs and Connectivity in Graphs

Subgraphs

- Let $\mathrm{G} = (V, E)$ be a simple graph;
- If $U \subseteq V$ and $F \subseteq E$, then $\mathrm{H} = (U, F)$ is a **subgraph** (podgraf) of G.
- If $U = V$, then H is a **spanning subgraph** (vpet podgraf) of G.
- If two vertices of H are adjacent in H whenever they are adjacent in G, then H is an **induced subgraph** (induciran podgraf).
- Note: An induced subgraph is fully determined by the set of vertices. Notation: $\mathrm{H} := \mathrm{G}[U]$.

Subgraphs - examples



Walks and Paths

- A *walk* (sprehod) in $\mathrm{G} = (V, E)$ is a sequence
$$W = [v_0, v_1, \ldots, v_k]$$
where $v_i \in V$ and $v_i \sim v_{i+1}$ for $i = 0, \ldots, v_k$.
  - v_0 is the beginning of W, v_k is the end of W.
  - W is a walk *from* v_0 *to* v_k (or between v_0 and v_k)
  - k is called *the length of* W (denoted by $\ell(W)$).
- Walk W is
  - *closed* (sklenjen) if $v_0 = v_k$,
  - *path* (pot) if v_i are all distinct,
  - *cycle* (cikel) if $v_0 = v_k$ and $v_1, \ldots, v_k$ are distinct.
- A path in G can also be viewed as a subgraph of G isomorphic to P_n.
- A cycle in G can also be viewed as a subgraph of G isomorphic to C_n.

Walks vs. Paths

Lemma

There is a walk between u and v in G if and only if there is a path between u and v in G.

Proof. Travel along the walk, and whenever you visit a vertex which has been visited before, cut out the part of the walk between the two occurrences of that same vertex.

Connectedness

- For $\mathrm{G} = (V, E)$, define a relation R on V:

$$uRv \Leftrightarrow \text{ there is a walk (path) between } u \text{ and } v.$$

- This relation is an equivalence relation (it is the transitive closure if the adjacency relation).
- Equivalence classes are **connected components** (povezane komponente) of G.
- G is **connected** (povezan) if it has only one connected component.

Distance in Graphs

- If G is connected, then $V = V(G)$ becomes a metric space for the following distance function:

$$\begin{aligned} d(u,v) &= \min\{\ell(W) : W \text{ is a walk between } u \text{ and } v\} \\ &= \text{the length of the shortest path between } u \text{ and } v. \end{aligned}$$

- The *diameter* (premer) of G is

$$\mathrm{diam}(\mathrm{G}) = \max\{d(u,v) : u, v \in V(\mathrm{G})\}.$$

Trees

A *tree* (drevo) is a graph in which for every pair of vertices there exists **exactly one** path between them.

Lemma

If G *is a graph, then the following are equivalent:*

1. G *is a tree (i.e.* G *is connected and there is exactly one path between any two vertices).*
2. G *is connected but removal of any edge disconnects it.*
3. G *is connected and* $|E(\mathrm{G})| = |V(\mathrm{G})| - 1$ *holds.*
4. G *contains no cycles and* $|E(\mathrm{G})| = |V(\mathrm{G})| - 1$ *holds.*
5. G *is connected and contains no cycles.*

Trees – the proof of characterization I

- The proof goes by induction on the number of vertices. The theorem clearly holds for graphs on 2 vertices. Suppose that it holds for all graphs on less than $|V(\mathrm{G})|$ vertices.
- $(1) \Rightarrow (2)$: Suppose G is a tree. Then it is connected by definition. Let $e = uv \in E(\mathrm{G})$. By assumption, $[u, v]$ is **the only** path between u and v. Hence, in $\mathrm{G} - e$ there is no path between u and v.

Trees – the proof of characterization II

- $(2) \Rightarrow (3)$: Assume that G is connected and removal of any edge disconnects it. We need to show that $|E(\mathrm{G})| = |V(\mathrm{G})| - 1$. Choose and edge e. By assumption, $\mathrm{G} - e$ is a union of (two) connected components X and Y. Clearly X and Y are connected graphs with the property that removal of any edge disconnects them. By induction, the implication $(2) \Rightarrow (3)$ holds for them, so $|E(X)| = |V(X)| - 1$ and $|E(Y)| = |V(Y)| - 1$. But then

$$|E(G)| = |E(X)| + |E(Y)| + 1 =$$

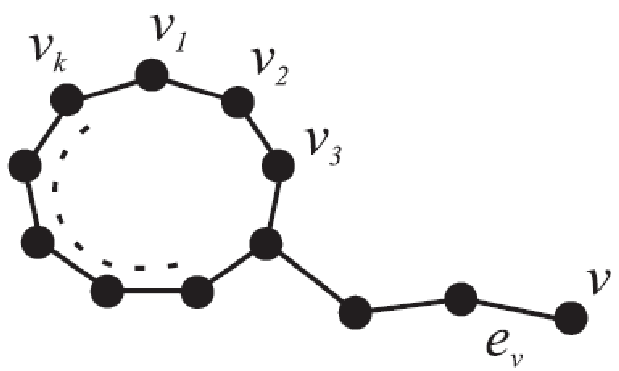
$$(|V(X)| - 1) + (|V(Y)| - 1) + 1 = |V(G)| - 1.$$

Trees – the proof of characterization III

- $(3) \Rightarrow (4)$: Suppose G is connected and $|E(\mathrm{G})| = |V(\mathrm{G})| - 1$ holds. We need to show that G has no cycles. Suppose on the contrary that C is a cycle. For any $v \notin V(C)$, let e_v be the first edge on the shortest path from v to C. Then

$$|E(\mathrm{G})| \geq |E(C)| \cup |\{e_v : v \in V(\mathrm{G}) \setminus V(C)\}| = V(\mathrm{G}),$$

a contradiction.



Trees – the proof of characterization IV

- $(4) \Rightarrow (5)$: Suppose G contains no cycles and that $|E(\mathrm{G})| = |V(\mathrm{G})| - 1$. We need to show that G is connected. Let $X_1, \ldots, X_k$ be connected components of G. Clearly, each X_i is connected and contains no cycles. By induction, this implies that X_i is a tree and that $|E(X_i)| = |V(X_i)| - 1$ holds. On the other hand

$$|E(\mathrm{G})| = \sum_{i=1}^{k} |E(X_i)| = \sum_{i=1}^{k} |V(X_i)| - k.$$

Since $|E(\mathrm{G})| = |V(\mathrm{G})| - 1$, we have $k = 1$ and so G is connected.

Trees – the proof of characterization V

- $(5) \Rightarrow (1)$: Suppose that G is connected and has no cycles. We need to show that there is exactly one path between any two points of G. By connectedness, there is at least one path. If there were two, then they would form a cycle. This completes the proof.

Trees

Corollary

A tree contains at least one vertex of valence 1.

Proof. Suppose that G is a graph in which every vertex has valence at least 2. By handshaking lemma,

$$|E(\mathrm{G})| = \frac{1}{2} \sum_{v \in V(\mathrm{G})} \mathrm{val}(v) \geq \frac{1}{2} |V(\mathrm{G})| \cdot 2 = |V(\mathrm{G})|.$$

On the other hand, if G is a tree, then

$$|E(\mathrm{G})| = |V(\mathrm{G})| - 1 < |V(\mathrm{G})|.$$

Hence G is not a tree. □

A vertex of valence 1 is called a *leaf* (list).

Spanning trees

A spanning subgraph which is a tree is called a *spanning tree* (vpeto drevo).

Lemma

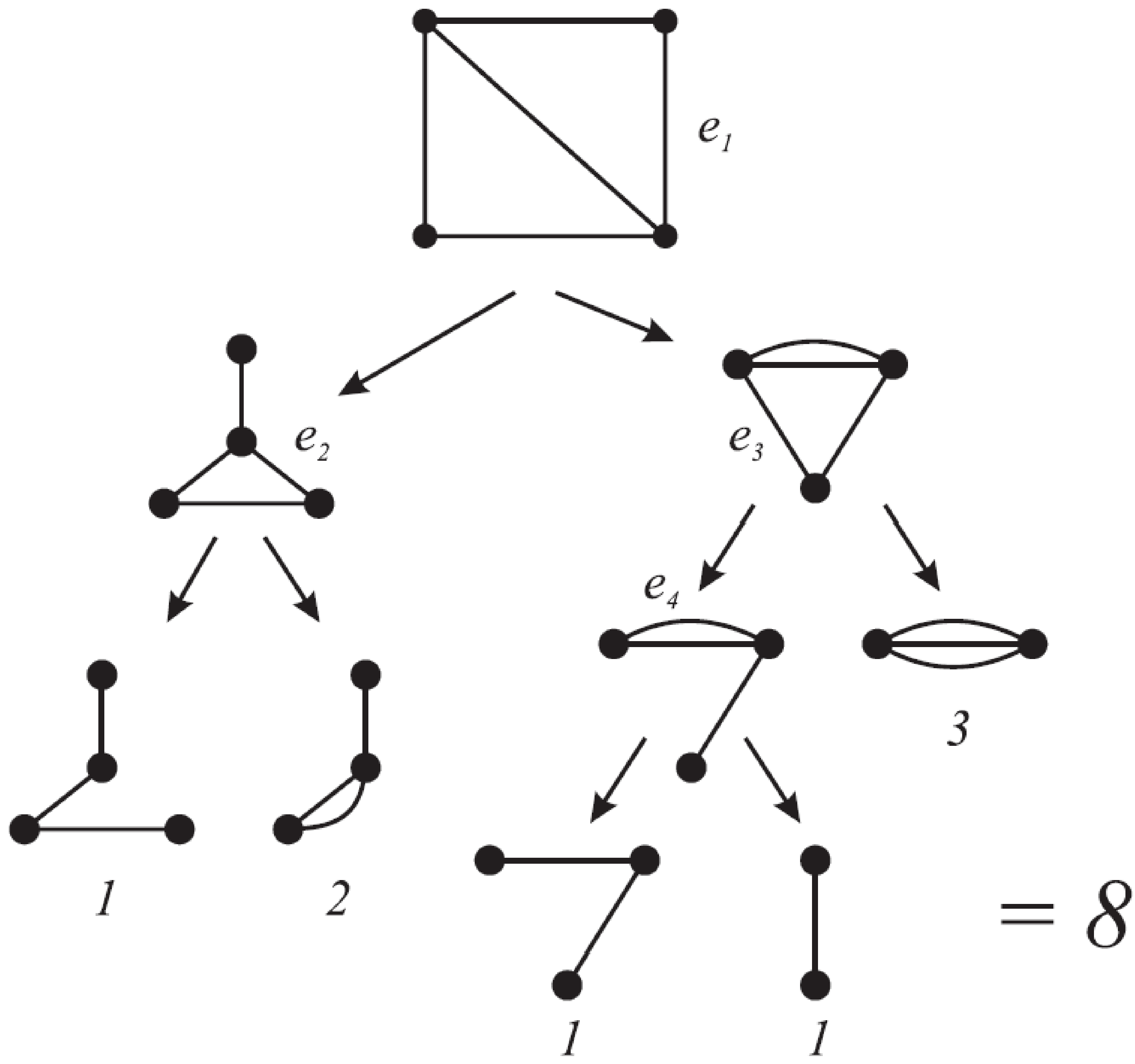
A graph G is connected if and only if contains a spanning tree.

Proof. The right-to-left direction is obvious. If G is connected and is not a tree, there exists a cycle C in it and we can remove an edge $e \in E(C)$ while preserving connectivity. Repeating this we get a subgraph of G - a spanning tree.

Number of spanning trees

- A connected graph may have more than one spanning tree.
- Cayley formula: $\tau(K_n) = n^{n-2}$.
- $\mathrm{G} - e$ ... *edge removal*: a graph obtained from G by removing the edge $e \in E(G)$ (but keeping in its end vertices)
- G/e ... *edge contraction*: remove e and merge the two end-vertices of $e \in E(G)$.
- In general: a recursive formula

$$\tau(\mathrm{G}) = \tau(\mathrm{G} - e) + \tau(G/e).$$



e1
e2
e3
e4
1
2
3
1
1
= 8

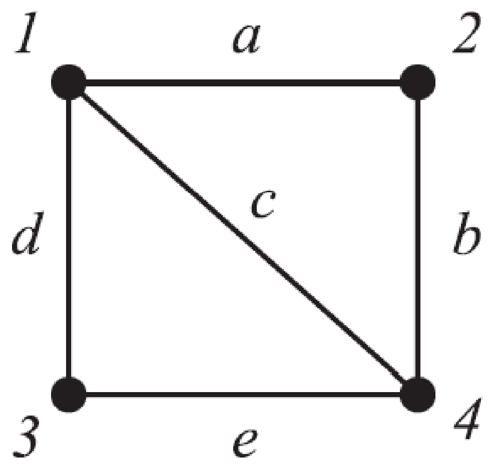
Laplacian matrix

- *Laplacian matrix* for a graph G is $L(\mathrm{G}) = D(\mathrm{G}) - A(\mathrm{G})$, where $D(\mathrm{G})$ denotes the diagonal matrix with entries $D_{i,i} = \mathrm{val}(v_i)$ and $V = \{v_1, \ldots, v_n\}$.
- Let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the non-zero eigenvalues of the Laplace matrix $L(\mathrm{G})$. Then

$$\tau(\mathrm{G}) = \frac{1}{n}\lambda_1\lambda_2\cdots\lambda_{n-1}.$$

- We get the same result if we calculate any cofactor of $L(G)$.

Example



$$L(G) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Any cofactor equals 8.

Spanning Paths and Cycles

- A spanning subgraph is also called a **factor**.
- A spanning path in a graph is also called a **hamilton path**.
- A spanning cycle in a graph is also called a **hamilton cycle**.

Isometric Subgraph

- $H=(U,F)$ is an isometric subgraph of graph $G=(V,E)$, if the distances are preserved:
- For each u,v 2 U: $d_H(u,v) = d_G(u,v)$.

Interval $I_G(u,v)$

- Let u, v 2 $V(G)$ belong to the same connected component of G. By $I_G(u,v)$ we denote the **interval** with endpoints u and v.
- $I_G(u,v)$ is the graph, induced on the set of vertices belonging to some shortest path from u to v.
- If there is no danger of confusion we can simplify notation: $I(u,v)$.

Convex Subgraph

- Graph H is a **convex subgraph** of G, if for every pair of vertices u and v from $V(H)$ that belong to the same connected component of G, the interval $I_G(u,v)$ is a subgraph of H.

k-connectedness

- Graph G with $|V(G)| > k$ is **k-connected**, if the removal of any set S with $|S| < k$ leaves a connected graph.
- **Connectivity** $\kappa(G)$ of graph G is the largest k, such that G is still k-connected.
- Vertex v of graph G is a **cut-vertex**, if $G - v$ contains more connected components than G.
- A connected graph with no cut-vertex is called a **block.**

2-connectedness

- **Theorem**: The following claims are equivalent:
  - Graph *G* is 2-connected,
  - Graph *G* is a block,
  - Any pair of vertices belongs to a common cycle

Menger’s Theorem

- Two paths in a graph with a common pair of end-vertices are **internally disjoint**, if they have no other vertex in common.
- **Theorem**: Graph is k-connected, if and only if there are k pair-wise internally disjoint paths between any two of its vertices.

Exercises

- N1. Show that if G has a hamilton cycle it also contains a hamilton path.
- N2. Show that every graph that has a hamilton path is connected.
- N3. Construct a graph on 10 vertices that has no hamilton path.
- N4. Construct a graph on 10 vertices that has no hamiloton cycle but has a hamilton path.
- N5: Construct a graph on 10 vertices that has a hamilton cycle.

Exercises 6-2

- N6. Determine all graphs with diameter 1.
- N7. Prove that each convex subgraph is an isometric subgraph.
- N8. Prove that each isometric subgraph is an induced subgraph.
- N9. Prove that each connected component is a convex subgraph.
- N10. Prove that the intersection of two induced subgraphs is an induced subgraph.
- N11. Prove that the intersection of two convex subgraphs is a convex subgraph.

Homework

- **H1.** Let C be the shortest cycle in graph G. Show that C is an induced subgraph of G.
- **H2**. Determine all non-isomorphic intervals in Q_4.
- **H3**. Find an isometric subgraph of Q_3 that is not convex.