EMERGENCE

SOME FEATURES:

Convergence to a consensus.

Local Averaging

(Decentralized) Mechanism

Learning (inductive)

Agents

Quantitative

THREE EXAMPLES:

Primitive Language

Flocking

Price Systems
Example: Origin of a primitive language. This was a joint work with F. Cucker and D-X. Zhou and can be found on my TTI-C website.

A language is a function \( f : M \rightarrow S \) where \( M \) is a finite set interpreted as a space of meanings and \( S \) is a convex set in Euclidean space interpreted as a space of sounds (and especially vowel sounds).

It is supposed that there are \( k \) linguistic agents, which we label as \( i \) going from 1 to \( k \).

The states of this system are denoted by

\[
(f_1, f_2, \ldots, f_k) \quad \text{in} \quad L \times L \times L \ldots \times L = L^k
\]

Where \( f_i \) is the language of agent \( i \) and \( L \) is the space of languages.
To describe a dynamics $T : L \xrightarrow{\kappa} L$ we use a $k$ by $k$ matrix of non-negative entries interpreted as the measure of linguistic encounters between agents and not necessarily symmetric. The matrix should have an irreducibility property. One could equivalently use a graph.

$T$ is defined by local averaging relative to the linguistic encounter matrix. Moreover the dynamics has a learning mechanism. Learning theory uses a set of examples which in this case amounts to learning a language. The examples are meaning-sound pairs $(m, s)$ in $M \times S$ emitted by each agent and transmitted by the encounter matrix to the listening agents.

Theorem. The dynamics starting with an arbitrary initial set of $k$ languages converges to a common language.

Remark. This result confirms numerical simulations of several linguists as Bill Wang.
MECHANISMS

Learning theory uses examples or data:

For $i = 1, \ldots, m$, $(x_i, y_i^*)$ in $X \times Y$, $Y = \mathbb{R}$ (the reals).

It is inductive learning.

Find an appropriate function $f : X \rightarrow Y$ which mimics $f(x_i^*) = y_i^*$.

Here is a version of curve fitting or interpolation.

Learning theory combines approximation theory and probability.

Goal: combine learning theory with local averaging to obtain a quantitative model of emergence, with universality.

Eventually one may obtain some universal laws of emergence.

Learning theory can supply a mechanism, as it relies on examples.

Moreover there exists a rather developed subject of learning theory in the last decades.
With Felipe Cucker,

**Flocking.** Birds $i = 1, \ldots, k$

Adjacency matrix $A = (a_{ij})$ where

$$a_{ij} = \frac{1}{(1 + \|x_i - x_j\|^2)^eta}, \quad x_i \in \mathbb{R}^3, \quad \beta \geq 0$$

"Laplacian" $L = D - A$, $D_{ii} = \sum_{j=1}^{k} a_{ij}$

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**Equations of flocking**

$X' = v$

$v' = -L v$

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Derived from

$$v_i(t+h) - v_i(t) = h \sum_{j=1}^{k} a_{ij} (v_j - v_i)$$

**Question.** Is there flocking?

Do solutions $v_i(t)$ converge to a common $v^* \in \mathbb{R}^3$?
Theorem for the equations of flocking, one has existence and uniqueness for all time.

If $\beta < \frac{1}{2}$, $v_i(t) \to v^* \in \mathbb{R}^3$ as $t \to \infty$, $v^*$ independent of $i$, and $x_i - x_j \to \hat{x}_{i,j}$ as $t \to \infty$, so that the relative positions stay bounded.

If $\beta \geq \frac{1}{2}$ dispersal is possible. But flocking as above will occur provided certain explicit initial conditions are satisfied.
Suppose that $G$ is a graph, defined by vertices and edges.

Let $A$ be its adjacency matrix so that the entries satisfy

$$a(i, j) = 1 \text{ if } (i, j) \text{ is an edge and zero otherwise.}$$

Let $D$ be the diagonal matrix defined by $d(i, i) = \sum_j a(i, j)$.

Then the Laplacian of $G$ is

$$L = L(G) = D - A$$

The eigenvalues of $L$ may be expressed by

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$$

The Fiedler number $F = F(G) = \lambda_2$

Fact: $F \neq 0$ if and only if $G$ is connected.

One extends these definitions to a general weighted graph

$a(i, j)$ (or such a matrix $A$).
For the flocking case $F$ depends on the positions $x$ and thus also on time. In fact if $F(x(t)) \geq \text{constant} > 0$ one obtains flocking; otherwise the birds disperse. This way the Fiedler number $F$ is a crucial invariant for the study of emergence.

In the above we may see an extension of the flocking model to cases where some $a(i,j)$ could be zero and one relaxes the condition of a complete graph.
The Fiedler number is inspired by the Cheeger Theorem in differential geometry, and the second eigenvalue of the Laplace-Beltrami operator.
For the proof of the flocking theorem it is useful to consider $x$ in $X$ where $X$ is the orthogonal space to the diagonal in the product space $\bigoplus_{i=1}^k (\mathbb{R}^3)^k$, and similarly $v$ in $V$ for the velocities.

Then the ordinary differential equations of flocking make sense in $X \times V$ and in this space flocking means $v(t) \rightarrow 0$ in $V$ and $x(t) \rightarrow x^*$ in $X$ as time goes to infinity.

Our conclusion in the flocking theorem above can be stated in these terms.
The first proposition in the proof is this.

Let the norm squared of $v(t)$ be denoted by $N$. Then

$$(d/dt)N(t) \leq (d/dt)N(0) \exp -t F^*(t)$$

where $F^*$ is the minimum of the Fiedler numbers $F(s)$ over $0 \leq s \leq t$.

The proof goes by $d/dt N = -<Lv, v> \leq - F N$.

Divide by $N$ and integrate etc.
FROM FLOCKING TO THE GENERAL EMERGENCE PROBLEM

Let \( X \) be some general Euclidean space.

Let \( V \) be the product space modulo the diagonal, just as before.

\[ L : X \rightarrow M(\kappa \times \kappa) \text{ (the \( \kappa \) by \( \kappa \) matrices) satisfy:} \]

\[
F(x) = \min_{v \in V} \frac{\langle L(x)v, v \rangle}{\langle v, v \rangle} \geq \frac{1}{(1 + \|x\|^2)^\beta}
\]

The differential equations are:

\[
\frac{dx}{dt} = f(x, v)
\]

\[
\frac{dv}{dt} = -L(x)v
\]

Here we suppose \( \|f(x, v)\| \leq K \|v\|^d \)

THEOREM. If \( \beta < \frac{1}{2} \) there is emergence.

ie. \( v(t) \rightarrow 0 \), etc., etc.
THE PRICE ADJUSTMENT PROBLEM

Playing the role of the previous velocity we use a corresponding

"Belief of a price system", or

function \( p \) from commodity space to a space of prices. Every
economic agent will have at a given time such a belief.

A learning mechanism is based on signals of the form

\((c_i, p_i)\) which could be interpreted as offers, bids, and even
exchanges.

This is very preliminary and not in the framework of general
equilibrium theory. A space of economic characteristics as
wealth, utility and economic relations will be our new \( X \).