Self-Organized Criticality

By

Chew Lock Yue, PhD

Division of Physics and Applied Physics
School of Physical and Mathematical Sciences
Nanyang Technological University
Who could ever calculate the path of a molecule? How do we know that the creation of worlds are not determined by falling grains of sand?

- Victor Hugo, *Les Misérables*
Self-Similarity

Road Network
Leaf Vascular Network
River Network

Lung Tubular Network
Blood Vessel Network
Neural Network
Fractals

$m = r^D$

$D = \frac{\ln m}{\ln r}$

$m \sim \text{Number of Copies}$

$r \sim \text{Scale Factor}$

$D \sim \text{Fractal Dimension}$

$m = 1$

$r = 1$

$m = 2$

$r = 2$

$m = 3$

$r = 3$

Koch Snowflake
Power Laws

\[ P(s) = s^{-\tau} \]

- Earthquake Magnitude
- Cotton Price
- Biological Extinction
- Ranking of Cities
- Ranking of Words
- X-Ray Intensities from Solar Flares
- Pulsar Glitches
- Earthquake Magnitude
1/f Noise

Intensity of Light from Quasar

Global Temperature
Criticality

Pressure

Liquid-gas critical point

\[ \epsilon = \frac{(T - T_C)}{T_C} \]

Density - \( \rho \)

Data for \( \text{CO}_2 \) and Xe. Critical index \( \beta \approx 0.34 \)

Ferromagnetism critical point

Specific Heat of MnF\(_2\). The power law \( C \sim \epsilon^{-0.16} \)

Universality

Ising Model

A model for ferromagnetic phase transition

\[ H = -J \sum_{i,j} \sigma_i \sigma_j - h \sum_i \sigma_i \]

Ising Model Simulation
\( (h = 0) \)

\[ T \to 0 \]

\[ T \to T_c \]

\[ T \to \infty \]
## Experimental Comparison

| Material               | Experimenters | Ref. | $T_e$ (°K)       | $\epsilon = |\Delta T| / T_e$ Range for fit | $\alpha$ |
|------------------------|---------------|------|------------------|-------------------------------|---------|
| **Antiferromagnets**   |               |      |                  |                               |         |
| MnF$_2$                | Teaney        | 86   | 67.33±0.01       | 2×10$^{-4}$–5×10$^{-2}$       | ≤0.16   |
| CoCl$_2$·6H$_2$O       | Skalyo, Friedberg | 84   | 2.289±0.002     | 10$^{-3}$–3×10$^{-2}$       | ≤0.11   |
| MnCl$_2$·4H$_2$O       | Friedberg, Wasscher | 89   | 1.622±0.005     | 10$^{-3}$–10$^{-1}$       |         |
| CuK$_2$(SO$_4$)$_2$·6H$_2$O | Miedema, Wielinga, Huiskamp | 85   | 0.193±0.001     | 10$^{-3}$–2×10$^{-2}$       | ≤0.6    |
| CoCs$_2$Cl$_6$         | Miedema, Wielinga, Huiskamp | 85   | 0.52±0.01       | 4×10$^{-1}$–2×10$^{-2}$     | ≤0.7    |
| RbMnF$_3$              | Teaney, Moruzzi, Argyle | 90   | 0.83±0.01       | 2×10$^{-1}$–5×10$^{-2}$     | ≤0.15   |
| **Ferromagnets**       |               |      |                  |                               |         |
| Iron                   | Kraftmakhér, Romashina | 91   | 1043.0±1.0      | 2×10$^{-3}$–10$^{-1}$       | ≤0.17   |
| CuK$_2$Cl$_4$·2H$_2$O  | Miedema, Wielinga, Huiskamp | 92   | 0.88±0.01       | 10$^{-3}$–10$^{-1}$       | ≤0.10   |
| Nickel                 | Kraftmakhér   | 93   | 627.0           | 5×10$^{-3}$–8×10$^{-2}$     |         |

Value used for scaling law analysis
Molecular field theory
3-dimensional Ising model

Specific Heat
# Experimental Comparison

| Material          | Experimenters        | Ref. | Method                        | $T_c$ (°K)    | $\epsilon = | \Delta T |/T_c$ Range for fit | $\beta$     |
|-------------------|----------------------|------|-------------------------------|---------------|----------------------|-------------|
| **Antiferromagnets** |                      |      |                               |               |                      |             |
| MnF$_2$           | Heller, Benedek      | 51   | NMR on F$^{19}$               | 67.336±0.003  | 8×10$^{-6}$–2×10$^{-2}$ | 0.335±0.01  |
| CuCl$_2$·2 H$_2$O | Poulis, Hardyman     | 76   | NMR, Protons                  | 4.337±0.003   | 5×10$^{-1}$–10$^{-2}$  | 0.18±0.07   |
| CoCl$_2$·6 H$_2$O | Sawatzky, Bloom      | 52   | NMR, Protons                  | 2.275         | 10$^{-4}$–10$^{-1}$    | 0.29±0.03   |
|                   | Van der Lugt, Poulis | 77   | NMR, Protons                  | 2.275         | 5×10$^{-1}$–2×10$^{-1}$ | 0.15±0.05   |
| KMnF$_3$          | Cooper, Nathans      | 69   | Neutron scattering            | 88.06±0.02    | 10$^{-2}$–10$^{-1}$    | 0.23±0.02   |
| **Ferromagnets**  |                      |      |                               |               |                      |             |
| Iron              | Preston, Hanna, Heberle | 71   | Mössbauer Fe$^{57}$           | 1042.0±0.3    | 2×10$^{-1}$–10$^{-1}$  | 0.34±0.02   |
|                   | Potter               | 78   | Magnetocaloric effect         | 1035.0±2.0    | 4×10$^{-1}$–2×10$^{-1}$ | 0.36±0.08   |
| Nickel            | Howard, Dunlap, Dash | 79   | Mössbauer Fe$^{57}$           | 629.4         | 10$^{-1}$–1.6×10$^{-1}$ | 0.51±0.04   |
|                   |                      |      | NMR, Eu$^{153}$              | 16.50±0.03    | 10$^{-2}$–10$^{-1}$    | 0.33±0.03   |
| EuS               | Heller, Benedek      | 79   | NMR, Eu$^{153}$              | 16.50±0.03    | 10$^{-2}$–10$^{-1}$    | 0.33±0.015  |
| YFeO$_2$          | Gorodetsky, Shtrikman, Treves | 30   | Vibrating sample magnetometer | 643           | 2×10$^{-3}$–3×10$^{-3}$ | 0.55±0.04   |
|                   | Ehrenfeucht, Shtrikman, Treves | 80   | Mössbauer Fe$^{57}$           | 640           | 10$^{-3}$–3×10$^{-3}$  | 0.354±0.005 |
| CrBr$_3$          | Senturia, Benedek    | 81   | NMR, Br$^{79}$, Br$^{81}$    | 32.56±0.015   | 7×10$^{-1}$–5×10$^{-2}$ | 0.365±0.015 |

Value used for scaling law analysis
Molecular field theory
3-dimensional Ising model
Complexity

- Higher Level Structures
- Lower Level Structures

Complexity

- Edge of Chaos
- Critical Point
  - Phase Transition
  - Order Parameter

- Regular
- Chaos

- Macroscopic Level
- Metastable, Complex Structures
- Microscopic Level
Coupled Pendulums

Per Bak
Pioneer in the physics of complex systems
Sandpiles

Discoverers of Self-Organized Criticality

Per Bak

Chao Tang

Kurt Wiesenfeld
Example of SOC: sandpile model on 2D square lattice (active) $z(x,y) → z(x,y) - 4$ (topple) $z(x \pm 1,y) → z(x\pm,y) + 1$ $z(x,y \pm 1) → z(x,y \pm 1) + 1$
Sandpile Dynamics

... sandpile model on 2D square lattice

| 1 | 1 | 0 | 2 | 3 | 0 |
| 0 | 3 | 2 | 2 | 2 | 3 |
| 2 | 2 | 2 | 1 | 0 | 2 |
| 2 | 0 | 0 | 3 | 1 | 1 |
| 1 | 1 | 3 | 4 | 2 | 1 |
| 3 | 2 | 1 | 1 | 0 | 2 |

| 1 | 1 | 0 | 2 | 3 | 0 |
| 0 | 3 | 2 | 2 | 2 | 3 |
| 2 | 2 | 2 | 1 | 0 | 2 |
| 2 | 0 | 1 | 4 | 1 | 1 |
| 1 | 2 | 0 | 1 | 3 | 1 |
| 3 | 2 | 2 | 2 | 0 | 2 |

| 1 | 1 | 0 | 2 | 3 | 0 |
| 0 | 3 | 2 | 2 | 2 | 3 |
| 2 | 2 | 2 | 1 | 0 | 2 |
| 2 | 0 | 1 | 0 | 2 | 1 |
| 1 | 1 | 4 | 1 | 3 | 1 |
| 3 | 2 | 1 | 2 | 0 | 2 |

| 1 | 1 | 0 | 2 | 3 | 0 |
| 0 | 3 | 2 | 2 | 2 | 3 |
| 2 | 2 | 2 | 1 | 0 | 2 |
| 2 | 0 | 2 | 0 | 2 | 1 |
| 1 | 2 | 0 | 2 | 3 | 1 |
| 3 | 2 | 2 | 2 | 0 | 2 |
Power Law Distributions

Distribution of Cluster Size

2-Dimensional

Distribution of Lifetime

3-Dimensional

Cluster Size

Lifetime
Self-Organized Criticality

Source: Netherlands Organization for Scientific Research

Fractals

Attractor for Metastable Configurations

1/f behavior
Self-Organized Criticality

Finite Size Scaling

\[ P(s) = \alpha_s s^{-\tau} G_s \left( \frac{s}{b_s L^D} \right) \]
SOC Features

- Slow Drive/Fast Relaxation
- Open/Dissipative
- Threshold/Instability
- Contingent/History
- Avalanche/Fluctuations
Experimental Verifications

Rotating Drum Experiment

IBM Experiment

Norwegian Rice Pile

University of Michigan Experiment
Earthquakes

OFC Model
Non-conservative SOC Model

\[ E_i \rightarrow E_i + \varepsilon \]

Homogeneous driving

\[ E_i \geq E_c \Rightarrow \begin{cases} E_i \rightarrow 0, \\ E_{nn} \rightarrow E_{nn} + \alpha E_i \end{cases} \]

Burridge-Knopoff Block-Spring Model

Gutenberg-Richter Law

The Earth Crust has self-organized to a critical state.
Bak-Sneppen Model: Random numbers between 0 and 1 are arranged in a circle. At each time step, the lowest number, and the number at its two neighbors, are each replaced by new random numbers.

\[ f(t) = f_c - A \left( \frac{t}{N} \right)^{-1/(\gamma-1)} \]

Self-Organization
Punctuated Equilibrium

- Cambrian Explosion
- Dinosaur Extinction

![Graphs and Diagrams]

- Phanerzoic Genera
- Power Law Behavior
- Devil's Staircase
- Thoracic Width

![Graphs and Diagrams]

- Accumulated Activity over time
- Mutation Activity
- Mean Thoracic Width with time markers

![Graphs and Diagrams]

- Extinction Percentages
- Geological Eras: Devonian, K-T and Triassic, Ordovician, Permian
**Forest Fire Model**

- A cell with burning tree turns into an empty cell
- A tree will burn if at least one neighbor is burning
- A tree ignites with probability $f$ even if no neighbor is burning
- A tree appears in an empty cell with probability $p$
Stock Market Crashes

- Imitation
- Herding
- Cooperativity
- Feedbacks

Instabilities

Self-Organizes to Criticality

Speculative Bubbles

Crashes

Graph showing a power law distribution with an exponent of -3, indicating

DJIA on Black Monday, 1987

Graph shows a spike in DJIA values during the month of October.
The Brain

Observation
Other thoughts

\{ \text{THOUGHTS} \} \sim \text{small or large avalanche}

Brain Self Organizes into a Critical State
- Subcritical \sim \text{access limited information}
- Supercritical \sim \text{too noisy}

\begin{align*}
\text{Signal} \\
(\text{Red or Green}) \\
\text{World}
\end{align*}
The critical state, with jams of all sizes, is the most efficient state, that can be reached dynamically.

Subcritical ~ free flow (under-utilization)

Supercritical ~ jammed (over-utilization)

From time series of number of vehicles at fixed location

Lifetime distribution from emergent jam

1/f noise

Power Spectral

Power Law
A Relook at SOC

- What is distinct about SOC?
  - Slowly driven, interaction dominated threshold system.
  - Self-organization versus tuning of parameters
  - Robustness of critical behavior

- Is there a theory of SOC systems?
  - Mean field theory
  - Exact solution in terms of operators for Abelian sandpile
  - Langevin equations
  - Dynamically driven renormalization group

- Has SOC taught us anything new about the world?
  - The importance of fluctuations

- Is there anyway predictive power in SOC?
  - Fluctuations have prevented us from predicting SOC systems in detail.
  - Understanding of mechanisms can provide insights into possible measures
  1. Having small or medium size fire/ Releasing social tensions in small or medium groups
  2. Create friction in the system ~ Cooling measures, e.g. Stock market, Property market.
Inconclusive experimental evidence on the possible causal relationship between the emergent power laws and the underlying self-organized critical state

- Variable selection
- Gibrat’s law – growth process by importance measure
- Coherent noise model (non-critical steady state)
- Highly optimized tolerance (non-critical self-organizing state)

Are the empirical distributions of complex systems exactly power law?

- Pareto, log-normal, log-Cauchy distributions look similar in log-log plot
- Heavy tailed distributions

Dragon Kings
References


References