Lesson No. 3: Graphs continued

1. Graph colorings
   - Vertex-colorings (Brook’s theorem, Mycielski’s construction)
   - Edge-colorings (Vizing’s theorem, König’s theorem, snarks)

2. Matchings (Hall’s theorem)
Let $G = (V, E)$ be a graph and $C$ a set of “colors”.

**Definition**

A **vertex-coloring** (barvanje točk) of $G$ is a function $c : V \rightarrow C$. The coloring is **proper** (pravilno) if $u \sim v \Rightarrow (v) \neq c(u)$. $G$ is $k$-vertex-colorable (točkovno $k$-obarvljiv) if there exists a proper vertex-coloring with $|C| = k$.

- $G$ is $k$-colorable $\Rightarrow$ $G$ is $\ell$-colorable for any $\ell \geq k$.
- **Chromatic number** (kormatično število) of $G$:

$$\chi(G) = \min\{k : G \text{ is } k\text{-vertex-colorable}\}$$
Vertex-colorings – examples

2-coloring of the cube.

3-coloring of the Petersen.
... more examples

- $\chi(K_n) = n$.
- $\chi(C_{2n}) = 2$.
- $\chi(C_{2n+1}) = 3$.
- If $H \leq G$, then $\chi(H) \leq \chi(G)$.

**Corollary**

*If $G$ contains a cycle of odd length, then $\chi(G) \geq 3$.***
Clearly, $\chi(G) = 1$ if and only if $G \cong K_n^C$.

**Lemma**

$\chi(G) \leq 2$ if and only if $G$ is bipartite.

**Proof:** ... think of color classes as bipartition sets ...

We know that graphs with $\chi \leq 2$ cannot have cycles of odd length. We will now show that the converse holds as well:

**Lemma**

*If $G$ contains no cycles of odd length, then $\chi(G) \leq 2$.***
Proof. WLOG: $G$ is connected. Choose $v \in V(G)$. For $u \in V(G)$ let:

- $c(u) = \text{“blue”}$ if $d(v, u)$ is even;
- $c(u) = \text{“red”}$ if $d(v, u)$ is odd.

If this is not a proper coloring, then there are two adjacent vertices $x, y$ that are both at even or both at odd distance from $v$.

Find shortest paths $P_x, P_y$ from $v$ to $x$ and to $y$. Then $P_x(xy)P_y^{-1}$ is a closed walk of odd length.

To complete the proof, we need to show the following:

**Exercise.** If a graph contains a closed walk of odd length, then it also contains a cycle of odd length.
This proves the following characterization of bipartite graphs.

**Theorem**

*If $G$ is a graph, then the following statements are equivalent:*

- $G$ is bipartite.
- $\chi(G) \leq 2$.
- $G$ contains no cycles of odd length.
Definition

A subset $U \subseteq V(G)$ is called a **clique** (klika), if the induced subgraph $G[U]$ is a complete graph.
**Definition**

A **maximal clique** (maksimalna klika) is a clique that is not contained in any other clique. A **largest clique** (največja klika) is a clique with the largest number of vertices among all cliques.

\[ \omega(G) = \text{“the size of the largest clique in } G\text{“}. \]

- Since \( \chi(K_n) = n \), it follows that \( \chi(G) \geq \omega(G) \).
The Brooks theorem

(Brooks) Let $G$ be a graph. Then

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1.$$ 

Moreover, $\chi(G) \leq \Delta(G)$ unless $G$ is a complete graph or a cycle of odd length.
Proof of the Brooks theorem

- WLOG: $G$ is connected.
- We already know that $\omega(G) \leq \chi(G)$. So we need to show two things:
  1. $\chi(G) \leq \Delta(G) + 1$.
  2. If $G \not\simeq K_n$ or $C_{2m+1}$, then $\chi(G) \leq \Delta(G)$.

Finding a $(\Delta + 1)$-coloring is easy:
- Let $\{1, \ldots, \Delta + 1\}$ be the set of colors. Order the vertices of $G$ in some linear order. Color the first vertex with color 1.
- Suppose that we have already colored the first $m$ vertices. Let $v$ be the next vertex, and let $c \in \{1, \ldots, \Delta + 1\}$ be the smallest integer that does not appear as a color of some neighbor of $v$. Color $v$ with the color $c$.
- Repeat this procedure until all vertices are colored.
Lecture 3 – Graphs continued

Graphs continued
Proof of the Brooks theorem – killing unessential greens

- In the rest of the proof, we may assume that \( G \not\cong K_n \) or \( C_{2m+1} \). It remains to show, that we may change the \((\Delta + 1)\)-coloring in such a way that one of the colors “disappears”.
- For the rest of the proof: \( \Delta + 1 = “green” \).
- Let \( S = “the set of green vertices” \).
- If there is \( v \in S \) such that one of the non-green colors does not appear among its neighbors, then we may use this color for \( v \). Apply this throughout \( S \). This procedure is called “killing unessential greens”.

- After unessential greens are “killed”, we get a \((\Delta + 1)\)-coloring in which each green vertex \( v \) has valence \( \Delta \), and no two neighbors of \( v \) are of the same color.
Proof of the Brooks theorem – pushing the green color

The second procedure allow us to “push” the green color from any \( v \in S \) to any other \( x \in V(G) \) along any path \( P \) from \( v \) to \( x \).

1. Kill unessential \((\Delta + 1)\)s. If \( S = \emptyset \) or \( S = \{x\} \), then stop.

2. Let \( u \) be the first vertex on the path from \( v \) to \( x \). Let \( c \) be the color of \( u \). Since we killed unessential greens, no green neighbours of \( u \) have any other neighbours of color \( c \).
Proof of the Brooks theorem – pushing the green color

1. Change the color of green neighbors of \( u \) to \( c \), and change the color of \( u \) to green.

2. Go to step 1 with \( u \) in place of \( v \), and with \( P \) being the part of old \( P \) from \( u \) to \( x \).

This procedure changed the color of some old green vertices, and cyclically rotated the colors along \( P \).
Let $x$ be a vertex of smallest valence. For each green $v$, choose a shortest path from $v$ to $x$, and push the green color to $x$. Now $x$ is the only green vertex.

If $\text{val}(x) < \Delta$, then there is a non-green color which does not appear among neighbors of $x$. Hence we can kill the green color at $x$, and finish.

It follows: We may thus assume that $G$ is regular (all vertices have valence $\Delta$).

The proof now splits into two cases:
- $G$ is 3-connected;
- $G$ is not 3-connected.
The proof of Brook’s theorem – the 3-connected case

- Suppose $G$ is 3-connected.
- Since $G$ is not complete, there exist $x \not\sim y$. Push the green color from all green vertices to $x$.
- Since there is no green color in $N(y)$, there exist $u, v \in N(y)$ of the same color.
Consider the graph $G' = G - u - v$. Since $G$ is 3-connected, $G'$ is connected. Choose a shortest path from $x$ to $y$ in $G'$ and push the green color from $x$ to $y$ along this path.
This results in a proper coloring of $G$ where the only green vertex is $y$, where $u$ and $v$ (two neighbors of $y$) have the same color. Therefore, the green color of $y$ can be “killed”, giving a $\Delta$-coloring of $G$. 
Suppose now that $G$ is not 3-connected.

The rest of the proof of is by induction on $n = |V(G)|$. By inspection, we see that the theorem holds for $n \leq 4$. Assume now that $n \geq 5$ and that theorem holds for all graphs with less than $n$ vertices.

If $\Delta(G) = 1$, then $G \cong K_n^C$, and so $\chi(G) = 1$.

If $\Delta(G) = 2$, then $G \cong C_n$ or $P_n$, and the theorem holds.

Assume henceforth that $\Delta(G) \geq 3$. 

Tomaž Pisanski, Alen Orbanič, and Primož Potočnik

Graphs continued
Suppose that $G$ has a cut-vertex $\{v\}$, and let $X_1, \ldots, X_m$ be the components of $G - v$.

By induction, each $X_i + v$ is $\Delta(G)$-colorable. By renaming colors in each $X_i$ if necessary, we may assume that in all $X_i$, the vertex $v$ has the same color. This gives a $\Delta(G)$-coloring of $G$. 
Suppose now that $G$ has a vertex-cut of size two: $\{x, y\}$.

In a similar way as in case $\kappa = 1$ we may use induction to show that $G$ is $\Delta$-colorable.

**Homework H2:** Finish the proof of the theorem in this case.
The Mycielski construction

- Let $G$ be a graph on $n$ vertices with at least one edge. Construct a new graph $G^+$ on $2n + 1$ vertices in the following way:
  - $V(G^+) = V(G) \cup \{v' : v \in V(G)\} \cup \{\infty\}$ (a disjoint union).
  - $E(G^+) = E(G) \cup \{v'u : vu \in E(G)\} \cup \{v'\infty : v \in V(G)\}$.
- Homework H3: Show that $\chi(G^+) = \chi(G) + 1$.
- Example: The graph, obtained in this way from $C_5$ is called the Grötzch graph.
Let $G = (V, E)$ be a graph and $C$ a set of “colors”. We define edge-colorings in a similar way as vertex-colorings:

**Definition**

An *edge-coloring* (barvanje povezav) of $G$ is a function $c: E \rightarrow C$. The coloring is *proper* if incident edges receive different colors. The graph $G$ is $k$-edge-colorable (povezavno $k$-obarvljiv) if there exists a proper edge-coloring with $|C| = k$.

- The minimal integer $k$ for which $G$ is $k$-edge-colorable is called the *chromatic index* (kormatični indeks) of $G$.

$$\chi'(G) = \min\{k : G \text{ is } k - \text{edge–colorable}\}$$

- Note that $\chi'(G) = \chi(L(G))$. 
Vizing’s theorem

- There is an obvious natural lower bound: $\chi'(G) \geq \Delta(G)$.
- The upper bound is given by Vizing’s theorem.

**Theorem**

\[
(Vizing) \quad \Delta(G) \leq \chi'(G) \leq \Delta(G) + 1
\]

- We skip the proof.
Graphs with $\chi'(G) = \Delta(G)$ are **graphs of class 1**, the others are of **class 2**.

- $C_{2m}$ is of class 1, $C_{2m+1}$ is of class 2.
- Hypercubes are of class 1
- The Petersen graph is of class 2.
- In general, determining $\chi'$ is difficult.
- For some graphs, this task is easier. For example, bipartite graphs.

**Theorem**

*(König) If $G$ is bipartite, then $\chi'(G) = \Delta(G)$.***
Proof of König’s theorem

- By contradiction: Let $k$ be a positive integer. Among all graphs with $\Delta = k$ choose a counterexample with the least number of edges.
- Choose an edge $e = xy$, such that $\Delta(G - e) = \Delta(G)$ (What if such an edge does not exist?).
- By hypothesis, $\chi'(G - e) = \Delta(G - e) = k$. Color the edges with $k$ colors.
Proof of König’s theorem II

There is a color $\alpha$, which does not appear at $x$, and a color $\beta$, which does not appear at $y$.

If $\alpha = \beta$, color $e$ with $\alpha$.

\begin{itemize}
  \item There is a color $\alpha$, which does not appear at $x$, and a color $\beta$, which does not appear at $y$.
  \item If $\alpha = \beta$, color $e$ with $\alpha$.
\end{itemize}
Assume now that $\alpha \neq \beta$.

Consider the subgraph $H$ induced by the edges of colors $\alpha$ and $\beta$. Clearly $\Delta(H) \leq 2$, so the connected components of $H$ are paths or cycles.
Note that swappings colors $\alpha$ and $\beta$ in any component of $H$ gives a different proper coloring.

Since all paths from $x$ to $y$ are of odd length ($G$ is bipartite!), $x$ and $y$ are in different components of $H$. Swap the colors $\alpha$ and $\beta$ in a component containing $x$. 
Finally, color \( e \) with \( \beta \).
A regular graph of valence 3 is called **cubic graph** (kubičen graf).

**Homework H3.** Show that every connected cubic graph with $\chi' = 3$ is 2-edge-connected.

On the other hand, it is not easy to find 2-edge-connected cubic graphs with $\chi' = 4$.

Such a graph is called a **snark**. (The name comes from a poem “The hunting of the Snark” by Lewis Carol.)

The smallest such graph is the Petersen graph.

Constructing new families of snarks is still a difficult task.
Matchings

Consider a proper edge-coloring of $G$. Consider the set $M$ of edges colored with a fixed color. No vertex of $G$ is incident with more than one edge from $M$.

**Definition**

A *matching* (prirejanje) in a graph $G$ is a set $M \subseteq E(G)$ such that each $v \in V(G)$ is incident with at most one $e \in M$.

- Vertices, that *are* incident with some $e \in M$ are saturated (nasičen).
- If every vertex of $G$ is saturated, then the matching is *perfect* (popolno prirejanje).
- A matching is *maximal* if it is the largest among all matching.
Maximal matchings are related to “stable sets”, “vertex covers” and “edge covers”

**Definition**

A **stable set** in $G$ is a set $U \subseteq V(G)$ such that no two vertices in $U$ are adjacent in $G$. A **vertex cover** in $G$ is a set $U \subseteq V(G)$ such that every edge of $G$ is incident with at least one vertex in $U$. An **edge cover** in $G$ is a set $F \subseteq E(G)$ such that every vertex of $G$ is incident with at least edge vertex in $F$.

- $\nu(G) := \text{“the size of a maximal matching } G\text{”}$;
- $\alpha(G) := \text{“the size of a largest stable set } G\text{”}$;
- $\tau(G) := \text{“the size of a smallest vertex cover of } G\text{”}$;
- $\rho(G) := \text{“the size of a smallest edge cover of } G\text{”}$. 
Gallai’s theorem and the König-Egerváry theorem

Theorem

(Gallai, 1959) If $G$ has no isolated vertices, then $
u(G) + \rho(G) = |V(G)|$.

Theorem

(König, Egerváry) If $G$ is bipartite, then $\nu(G) = \tau(G)$.

We skip the proofs.
It is often difficult to decide, what is the size of a largest matching. For bipartite graphs, we have the following nice result:

**Theorem (Hall)** Let $G$ be a bipartite graph with bipartition $V(G) = X \cup Y$. Then $G$ has a matching in which every vertex of $X$ is saturated if and only if $|N(S)| \geq |S|$ for every set $S \subseteq X$.

Here $N(S)$ is the set of vertices that are adjacent to some vertex in $S$. 
Proof of Hall’s theorem

- **Proof.** One direction is obvious.
- For the other direction, we need the König-Egerváry theorem. Suppose that there is no matching in which every \( v \in X \) is saturated. Then \( \nu(G) < |X| \).
- By the König-Egerváry theorem, \( \nu(G) = \tau(G) \). Therefore, there is a vertex cover \( K \) with \( |K| < |X| \).
- Let \( S = X \setminus K \). Then \( N(S) \subseteq Y \cap K \), and so

\[
|S| = |X| - |K \cap X| = |X| - |K| + |Y \cap K| > N(S).
\]
Homework

H1 Finish the proof of the Brooks theorem in the case where the vertex-connectivity of the graph is 2.

H2 Show that \( \chi(G^+) = \chi(G) + 1 \).

H3 Show that every connected cubic graph with \( \chi' = 3 \) is 2-edge-connected.