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**Dynamic portfolio selection: general case**

\[ x_i = (x_i^{(1)}, \ldots, x_i^{(d)}) \] the return vector on day \( i \)

\( b = b_1 \) is the portfolio vector for the first day

initial capital \( S_0 \)

\[ S_1 = S_0 \cdot \langle b_1, x_1 \rangle \]
Dynamic portfolio selection: general case

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initial capital \( S_0 \)

\[
S_1 = S_0 \cdot \langle \mathbf{b}_1 , \mathbf{x}_1 \rangle
\]

for the second day, \( S_1 \) new initial capital, the portfolio vector

\( \mathbf{b}_2 = \mathbf{b}(\mathbf{x}_1) \)

\[
S_2 = S_0 \cdot \langle \mathbf{b}_1 , \mathbf{x}_1 \rangle \cdot \langle \mathbf{b}(\mathbf{x}_1) , \mathbf{x}_2 \rangle.
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Dynamic portfolio selection: general case

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nth day a portfolio strategy \( b_n = b(x_1, \ldots, x_{n-1}) = b(x_1^{n-1}) \)

\[ S_n = S_0 \prod_{i=1}^{n} \langle b(x_1^{i-1}), x_i \rangle = S_0 e^{nW_n(B)} \]

with the average growth rate

\[ W_n(B) = \frac{1}{n} \sum_{i=1}^{n} \ln \langle b(x_1^{i-1}), x_i \rangle. \]
\( X_1, X_2, \ldots \) drawn from the vector valued stationary and ergodic process

log-optimum portfolio \( B^* = \{b^*(\cdot)\} \)

\[
\mathbb{E}\{\ln \langle b^*(X_1^{n-1}), X_n \rangle \mid X_1^{n-1}\} = \max_{b(\cdot)} \mathbb{E}\{\ln \langle b(X_1^{n-1}), X_n \rangle \mid X_1^{n-1}\}
\]

\( X_1^{n-1} = X_1, \ldots, X_{n-1} \)
Algoet and Cover (1988): If \( S_n^* = S_n(B^*) \) denotes the capital after day \( n \) achieved by a log-optimum portfolio strategy \( B^* \), then for any portfolio strategy \( B \) with capital \( S_n = S_n(B) \) and for any process \( \{X_n\}_{-\infty}^{\infty} \),

\[
\limsup_{n \to \infty} \left( \frac{1}{n} \ln S_n - \frac{1}{n} \ln S_n^* \right) \leq 0 \quad \text{almost surely}
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Algoet and Cover (1988): If $S_n^* = S_n(B^*)$ denotes the capital after day $n$ achieved by a log-optimum portfolio strategy $B^*$, then for any portfolio strategy $B$ with capital $S_n = S_n(B)$ and for any process $\{X_n\}_{-\infty}^{\infty}$,

$$\limsup_{n \to \infty} \left( \frac{1}{n} \ln S_n - \frac{1}{n} \ln S_n^* \right) \leq 0 \quad \text{almost surely}$$

for stationary ergodic process $\{X_n\}_{-\infty}^{\infty}$,

$$\lim_{n \to \infty} \frac{1}{n} \ln S_n^* = W^* \quad \text{almost surely},$$

where

$$W^* = \mathbb{E}\left\{ \max_{b(\cdot)} \mathbb{E}\{ \ln \langle b(X^{-1}_{-\infty}) , X_0 \rangle \mid X^{-1}_{-\infty} \} \right\}$$

is the maximal growth rate of any portfolio.
for the proof of optimality we use the concept of martingale differences:

**Definition**

there are two sequences of random variables:

\[
\{Z_n\} \quad \{X_n\}
\]

- \(Z_n\) is a function of \(X_1, \ldots, X_n\),
- \(E\{Z_n \mid X_1, \ldots, X_{n-1}\} = 0\) almost surely.

Then \(\{Z_n\}\) is called martingale difference sequence with respect to \(\{X_n\}\).
Chow Theorem: If \( \{Z_n\} \) is a martingale difference sequence with respect to \( \{X_n\} \) and

\[
\sum_{n=1}^{\infty} \frac{\mathbb{E}\{Z_n^2\}}{n^2} < \infty
\]

then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} Z_i = 0 \text{ a.s.}
\]
A weak law of large numbers

**Lemma:** If \( \{Z_n\} \) is a martingale difference sequence with respect to \( \{X_n\} \) then \( \{Z_n\} \) are uncorrelated.

**Proof.** Put \( i < j \).

\[
E \{Z_i Z_j\} = E \{E \{Z_i Z_j | X_1, \ldots, X_{j-1}\}\} = E \{Z_i E \{Z_j | X_1, \ldots, X_{j-1}\}\} = E \{Z_i \cdot 0\} = 0
\]

**Corollary**

\[
\frac{1}{n} \sum_{i=1}^{n} Z_i^2 \rightarrow 0
\]

if, for example, \( E \{Z_i^2\} \) is a bounded sequence.
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Corollary

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E \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} Z_i \right)^2 \right\}
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E \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} Z_i \right)^2 \right\} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} E\{Z_i Z_j\}
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E\left\{\left(\frac{1}{n} \sum_{i=1}^{n} Z_i\right)^2\right\} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} E\{Z_i Z_j\} \\
= \frac{1}{n^2} \sum_{i=1}^{n} E\{Z_i^2\} \\
\rightarrow 0
\]

if, for example, \( E\{Z_i^2\} \) is a bounded sequence.
Constructing martingale difference sequence

\{ Y_n \} is an arbitrary sequence such that \( Y_n \) is a function of \( X_1, \ldots, X_n \)
Constructing martingale difference sequence

\{Y_n\} is an arbitrary sequence such that \(Y_n\) is a function of \(X_1, \ldots, X_n\).

Put

\[ Z_n = Y_n - \mathbb{E}\{Y_n \mid X_1, \ldots, X_{n-1}\} \]

Then \(\{Z_n\}\) is a martingale difference sequence:
Constructing martingale difference sequence

\{Y_n\} is an arbitrary sequence such that \(Y_n\) is a function of \(X_1, \ldots, X_n\)

Put

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Z_n = Y_n - \mathbb{E}\{ Y_n \mid X_1, \ldots, X_{n-1} \}
\]

Then \( \{ Z_n \} \) is a martingale difference sequence:

- \( Z_n \) is a function of \( X_1, \ldots, X_n \),

- \[
\mathbb{E}\{ Z_n \mid X_1, \ldots, X_{n-1} \} = \mathbb{E}\{ Y_n \mid X_1, \ldots, X_{n-1} \} = 0
\] almost surely.
Constructing martingale difference sequence

\{Y_n\} is an arbitrary sequence such that \(Y_n\) is a function of \(X_1, \ldots, X_n\)

Put

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= 0
\]

almost surely.
log-optimum portfolio $B^* = \{b^*(\cdot)\}$

$$E\{\ln \langle b^*(X_1^{n-1}), X_n \rangle | X_1^{n-1} \} = \max_{b(\cdot)} E\{\ln \langle b(X_1^{n-1}), X_n \rangle | X_1^{n-1} \}$$
Optimality

log-optimum portfolio $B^* = \{b^*(\cdot)\}$

$$E\{\ln \langle b^*(X_1^{n-1}) , X_n \rangle \mid X_1^{n-1} \} = \max_{b(\cdot)} E\{\ln \langle b(X_1^{n-1}) , X_n \rangle \mid X_1^{n-1} \}$$

If $S_n^* = S_n(B^*)$ denotes the capital after day $n$ achieved by a log-optimum portfolio strategy $B^*$, then for any portfolio strategy $B$ with capital $S_n = S_n(B)$ and for any process $\{X_n\}_{-\infty}^{\infty}$,

$$\limsup_{n \to \infty} \left( \frac{1}{n} \ln S_n - \frac{1}{n} \ln S_n^* \right) \leq 0 \quad \text{almost surely}$$
Proof of optimality

\[
\frac{1}{n} \ln S_n = \frac{1}{n} \sum_{i=1}^{n} \ln \left\langle b\left(\mathbf{X}_{1}^{i-1}\right), \mathbf{X}_i \right\rangle
\]
Proof of optimality

\[
\frac{1}{n} \ln S_n = \frac{1}{n} \sum_{i=1}^{n} \ln \left< b(X_{i-1}^i), X_i \right>
\]

= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \{ \ln \left< b(X_{1}^{i-1}), X_i \right> | X_{1}^{i-1} \}

+ \frac{1}{n} \sum_{i=1}^{n} \left( \ln \left< b(X_{1}^{i-1}), X_i \right> - \mathbb{E} \{ \ln \left< b(X_{1}^{i-1}), X_i \right> | X_{1}^{i-1} \} \right)
Proof of optimality

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\frac{1}{n} \ln S_n = \frac{1}{n} \sum_{i=1}^{n} \ln \left< b(X_{i-1}^i), X_i \right>
\]

\[
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\]

\[
+ \frac{1}{n} \sum_{i=1}^{n} \left( \ln \left< b(X_{1-1}^i), X_i \right> - \mathbb{E} \{ \ln \left< b(X_{1-1}^i), X_i \right> | X_{1-1}^i \} \right)
\]

and

\[
\frac{1}{n} \ln S_n^* = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \{ \ln \left< b^*(X_{1-1}^i), X_i \right> | X_{1-1}^i \} 
\]

\[
+ \frac{1}{n} \sum_{i=1}^{n} \left( \ln \left< b^*(X_{1-1}^i), X_i \right> - \mathbb{E} \{ \ln \left< b^*(X_{1-1}^i), X_i \right> | X_{1-1}^i \} \right)
\]
These limit relations give rise to the following definition:

**Definition**

An empirical (data driven) portfolio strategy $B$ is called **universally consistent with respect to a class $C$ of stationary and ergodic processes** $\{X_n\}_{-\infty}^{\infty}$, if for each process in the class,

$$\lim_{n \to \infty} \frac{1}{n} \ln S_n(B) = W^*$$

almost surely.
\[ E\{\ln \langle b^*(X_1^{n-1}), X_n \rangle \mid X_1^{n-1} \} = \max_{b(\cdot)} E\{\ln \langle b(X_1^{n-1}), X_n \rangle \mid X_1^{n-1} \} \]
Empirical portfolio selection

\[ E\{ \ln \langle b^*(X_1^{n-1}), X_n \rangle \mid X_1^{n-1} \} = \max_{b(\cdot)} E\{ \ln \langle b(X_1^{n-1}), X_n \rangle \mid X_1^{n-1} \} \]

fixed integer \( k > 0 \)

\[ E\{ \ln \langle b(X_1^{n-1}), X_n \rangle \mid X_1^{n-1} \} \approx E\{ \ln \langle b(X_{n-k}^{n-1}), X_n \rangle \mid X_{n-k}^{n-1} \} \]
Empirical portfolio selection

\[ E\{\ln \langle b^*(X_{n-1}^1), X_n \rangle \mid X_{n-1}^1 \} = \max_{b(\cdot)} E\{\ln \langle b(X_{n-1}^1), X_n \rangle \mid X_{n-1}^1 \} \]

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and

\[ b^*(X_{n-1}^1) \approx b_k(X_{n-k}^{n-1}) = \arg \max_{b(\cdot)} E\{\ln \langle b(X_{n-k}^{n-1}), X_n \rangle \mid X_{n-k}^{n-1} \} \]
\[ E\{\ln \langle b^*(X_{1}^{n-1}) , X_n \rangle | X_{1}^{n-1} \} = \max_{b(\cdot)} E\{\ln \langle b(X_{1}^{n-1}) , X_n \rangle | X_{1}^{n-1} \} \]

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and

\[ b^*(X_{1}^{n-1}) \approx b_k(X_{n-k}^{n-1}) = \arg \max_{b(\cdot)} E\{\ln \langle b(X_{n-k}^{n-1}) , X_n \rangle | X_{n-k}^{n-1} \} \]

because of stationarity

\[ b_k(x_1^k) = \arg \max_{b(\cdot)} E\{\ln \langle b(x_1^k) , X_{k+1} \rangle | X_1^k = x_1^k \} \]

\[ = \arg \max_{b} E\{\ln \langle b , X_{k+1} \rangle | X_1^k = x_1^k \}, \]
Empirical portfolio selection

\[ E\{\ln \langle b^* (X_1^{n-1}), X_n \rangle \mid X_1^{n-1}\} = \max_{b(\cdot)} E\{\ln \langle b(X_1^{n-1}), X_n \rangle \mid X_1^{n-1}\} \]

fixed integer \( k > 0 \)

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\[ b^* (X_1^{n-1}) \approx b_k (X_{n-k}^{n-1}) = \arg \max_{b(\cdot)} E\{\ln \langle b(X_{n-k}^{n-1}), X_n \rangle \mid X_{n-k}^{n-1}\} \]

because of stationarity

\[ b_k (x_1^k) = \arg \max_{b(\cdot)} E\{\ln \langle b(x_1^k), X_{k+1} \rangle \mid X_1^k = x_1^k\} \]

\[ = \arg \max_b E\{\ln \langle b, X_{k+1} \rangle \mid X_1^k = x_1^k\}, \]

which is the maximization of the regression function

\[ m_b (x_1^k) = E\{\ln \langle b, X_{k+1} \rangle \mid X_1^k = x_1^k\} \]
Regression function

\( Y \) real valued
\( X \) observation vector
Regression function

$Y$ real valued
$X$ observation vector
Regression function

\[ m(x) = \mathbb{E}\{Y \mid X = x\} \]

i.i.d. data: $D_n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$
Regression function

$Y$ real valued
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i.i.d. data: \( D_n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\} \)
Regression function estimate

\[ m_n(x) = m_n(x, D_n) \]
Regression function

\( Y \) real valued
\( X \) observation vector

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m(x) = \mathbb{E}\{ Y \mid X = x \}
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Regression function estimate

\[
m_n(x) = m_n(x, D_n)
\]

local averaging estimates

\[
m_n(x) = \sum_{i=1}^{n} W_{ni}(x; X_1, \ldots, X_n) Y_i
\]
Regression function

\( Y \) real valued
\( X \) observation vector
Regression function

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Correspondence

\[ x \quad x^k_1 \]
\begin{align*}
X & \quad X_1^k \\
Y & \quad \ln \langle b, X_{k+1} \rangle
\end{align*}
\[ X \quad X_1^k \]
\[ Y \quad \ln \langle b, X_{k+1} \rangle \]
\[ m(x) = \mathbb{E}\{Y \mid X = x\} \quad m_b(x_1^k) = \mathbb{E}\{\ln \langle b, X_{k+1} \rangle \mid X_1^k = x_1^k\} \]
Partition $\mathcal{P}_n = \{A_{n,1}, A_{n,2} \ldots \}$
Partitioning regression estimate

Partition $\mathcal{P}_n = \{A_{n,1}, A_{n,2} \ldots \}$

$A_n(x)$ is the cell of the partition $\mathcal{P}_n$ into which $x$ falls
Partitioning regression estimate

Partition $\mathcal{P}_n = \{A_{n,1}, A_{n,2}, \ldots \}$

$A_n(x)$ is the cell of the partition $\mathcal{P}_n$ into which $x$ falls

$$m_n(x) = \frac{\sum_{i=1}^{n} Y_i I[X_i \in A_n(x)]}{\sum_{i=1}^{n} I[X_i \in A_n(x)]}$$
Partitioning regression estimate

Partition $\mathcal{P}_n = \{A_{n,1}, A_{n,2} \ldots \}$

$A_n(x)$ is the cell of the partition $\mathcal{P}_n$ into which $x$ falls

$$m_n(x) = \frac{\sum_{i=1}^{n} Y_i I[X_i \in A_n(x)]}{\sum_{i=1}^{n} I[X_i \in A_n(x)]}$$

Let $G_n$ be the quantizer corresponding to the partition $\mathcal{P}_n$:

$G_n(x) = j$ if $x \in A_{n,j}$. 
Partition \( \mathcal{P}_n = \{A_{n,1}, A_{n,2} \ldots \} \)

\( A_n(x) \) is the cell of the partition \( \mathcal{P}_n \) into which \( x \) falls

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the set of matches

\[
I_n = \{i \leq n: G_n(x) = G_n(X_i)\}
\]

Then

\[
m_n(x) = \frac{\sum_{i \in I_n} Y_i}{|I_n|}.
\]
Partitioning-based portfolio selection

fix \( k, \ell = 1, 2, \ldots \)

\( \mathcal{P}_\ell = \{ A_{\ell,j}, j = 1, 2, \ldots, m_\ell \} \) finite partitions of \( \mathbb{R}^d \),
fix $k, \ell = 1, 2, \ldots$
\[ \mathcal{P}_\ell = \{ A_{\ell,j}, j = 1, 2, \ldots, m_\ell \} \] finite partitions of $\mathbb{R}^d$,
$G_\ell$ be the corresponding quantizer: $G_\ell(x) = j$, if $x \in A_{\ell,j}$. 

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$G_\ell(x_1^n) = G_\ell(x_1), \ldots, G_\ell(x_n)$,
Partitioning-based portfolio selection

fix \( k, \ell = 1, 2, \ldots \)

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\( G_\ell \) be the corresponding quantizer: \( G_\ell(x) = j \), if \( x \in A_{\ell,j} \).

\( G_\ell(x^n_1) = G_\ell(x_1), \ldots, G_\ell(x_n) \),

the set of matches:

\[
J_n = \{ k < i < n : G_\ell(x^{i-1}_{i-k}) = G_\ell(x^{n-1}_{n-k}) \}
\]
fix \( k, \ell = 1, 2, \ldots \)

\[ \mathcal{P}_\ell = \{ A_{\ell,j}, j = 1, 2, \ldots, m_\ell \} \] finite partitions of \( \mathbb{R}^d \),

\( G_\ell \) be the corresponding quantizer: \( G_\ell(x) = j \), if \( x \in A_{\ell,j} \).

\( G_\ell(x^n_1) = G_\ell(x_1), \ldots, G_\ell(x_n) \),

the set of matches:

\[ J_n = \{ k < i < n : G_\ell(x_{i-1}) = G_\ell(x_{n-1}) \} \]

\[ b^{(k,\ell)}(x_1^{n-1}) = \arg \max_b \sum_{i \in J_n} \ln \langle b, x_i \rangle \]

if the set \( I_n \) is non-void, and \( b_0 = (1/d, \ldots, 1/d) \) otherwise.
for fixed \( k, \ell = 1, 2, \ldots, \)
\[ \mathbf{B}(k,\ell) = \{ \mathbf{b}(k,\ell)(\cdot) \}, \] are called elementary portfolios
for fixed $k, \ell = 1, 2, \ldots,$
$B^{(k,\ell)} = \{b^{(k,\ell)}(\cdot)\}$, are called elementary portfolios

That is, $b^{(k,\ell)}_n$ quantizes the sequence $x_{1:n-1}$ according to the partition $P_\ell$, and browses through all past appearances of the last seen quantized string $G_\ell(x_{n-k}^{n-1})$ of length $k$.
Then it designs a fixed portfolio vector according to the returns on the days following the occurrence of the string.
How to choose $k, \ell$

- small $k$ or small $\ell$: large bias
- large $k$ and large $\ell$: few matching, large variance
Combining elementary portfolios

How to choose $k, \ell$

- small $k$ or small $\ell$: large bias
- large $k$ and large $\ell$: few matching, large variance

Machine learning: combination of experts

Exponential weighing

combine the elementary portfolio strategies $B^{(k,\ell)} = \{b_n^{(k,\ell)}\}$
Exponential weighing

combine the elementary portfolio strategies $B^{(k,\ell)} = \{b^{(k,\ell)}\}$
let $\{q_{k,\ell}\}$ be a probability distribution on the set of all pairs $(k, \ell)$
such that for all $k, \ell$, $q_{k,\ell} > 0$. 
Exponential weighing

combine the elementary portfolio strategies \( B^{(k,\ell)} = \{ b_n^{(k,\ell)} \} \)

let \( \{ q_{k,\ell} \} \) be a probability distribution on the set of all pairs \((k, \ell)\)
such that for all \( k, \ell, q_{k,\ell} > 0 \).

for \( \eta > 0 \) put

\[
w_{n,k,\ell} = q_{k,\ell} e^{\eta \ln S_{n-1}(B^{(k,\ell)})}
\]
Exponential weighing

combine the elementary portfolio strategies $B^{(k,\ell)} = \{b^{(k,\ell)}_n\}$

let $\{q_{k,\ell}\}$ be a probability distribution on the set of all pairs $(k, \ell)$ such that for all $k, \ell$, $q_{k,\ell} > 0$.

for $\eta > 0$ put

$$w_{n,k,\ell} = q_{k,\ell} e^{\eta \ln S_{n-1}(B^{(k,\ell)})}$$

for $\eta = 1$,

$$w_{n,k,\ell} = q_{k,\ell} e^{\ln S_{n-1}(B^{(k,\ell)})} = q_{k,\ell} S_{n-1}(B^{(k,\ell)})$$
Exponential weighing

combine the elementary portfolio strategies \( \mathbf{B}^{(k,\ell)} = \{ \mathbf{b}^{(k,\ell)}_n \} \)

let \( \{ q_k,\ell \} \) be a probability distribution on the set of all pairs \((k, \ell)\)
such that for all \( k, \ell, q_k,\ell > 0 \).

for \( \eta > 0 \) put

\[
\frac{1}{\eta} \ln S_{n-1}^{(k,\ell)}
\]

for \( \eta = 1, \)

\[
1 \ln S_{n-1}^{(k,\ell)}
\]

and

\[
\mathbf{v}_{n,k,\ell} = \frac{w_{n,k,\ell}}{\sum_{i,j} w_{n,i,j}}.
\]
Exponential weighing

combine the elementary portfolio strategies \( \mathbf{B}(k, \ell) = \{ \mathbf{b}_n^{(k, \ell)} \} \)

let \( \{ q_{k, \ell} \} \) be a probability distribution on the set of all pairs \( (k, \ell) \)
such that for all \( k, \ell, \) \( q_{k, \ell} > 0. \)

for \( \eta > 0 \) put

\[
w_{n, k, \ell} = q_{k, \ell} e^{\eta \ln S_{n-1}(\mathbf{B}(k, \ell))}
\]

for \( \eta = 1, \)

\[
w_{n, k, \ell} = q_{k, \ell} e^{\ln S_{n-1}(\mathbf{B}(k, \ell))} = q_{k, \ell} S_{n-1}(\mathbf{B}(k, \ell))
\]

and

\[
v_{n, k, \ell} = \frac{w_{n, k, \ell}}{\sum_{i,j} w_{n, i,j}}.
\]

the combined portfolio \( \mathbf{b} \):

\[
\mathbf{b}_n(x_1^{n-1}) = \sum_{k,\ell} v_{n, k, \ell} \mathbf{b}_n^{(k, \ell)}(x_1^{n-1}).
\]
\[ S_n(B) = \prod_{i=1}^{n} \langle b_i(x_1^{i-1}), x_i \rangle \]
\[
S_n(B) = \prod_{i=1}^{n} \langle b_i(x_1^{i-1}), x_i \rangle \\
= \prod_{i=1}^{n} \frac{\sum_{k, \ell} w_{i, k, \ell} \langle b_i^{(k, \ell)} (x_1^{i-1}), x_i \rangle}{\sum_{k, \ell} w_{i, k, \ell}}
\]
\[ S_n(B) = \prod_{i=1}^{n} \left\langle b_i(x_{i-1}^i), x_i \right\rangle \]

\[ = \prod_{i=1}^{n} \frac{\sum_{k,\ell} w_{i,k,\ell} \left\langle b_i^{(k,\ell)}(x_{i-1}^i), x_i \right\rangle}{\sum_{k,\ell} w_{i,k,\ell}} \]

\[ = \prod_{i=1}^{n} \frac{\sum_{k,\ell} q_{k,\ell} S_{i-1}(B^{(k,\ell)}) \left\langle b_i^{(k,\ell)}(x_{i-1}^i), x_i \right\rangle}{\sum_{k,\ell} q_{k,\ell} S_{i-1}(B^{(k,\ell)})} \]
\[ S_n(B) = \prod_{i=1}^{n} \left\langle b_i(x_1^{i-1}), x_i \right\rangle \]

\[ = \prod_{i=1}^{n} \frac{\sum_{k, \ell} w_{i, k, \ell} \left\langle b_i^{(k, \ell)}(x_1^{i-1}), x_i \right\rangle}{\sum_{k, \ell} w_{i, k, \ell}} \]

\[ = \prod_{i=1}^{n} \frac{\sum_{k, \ell} q_{k, \ell} S_{i-1}(B^{(k, \ell)}) \left\langle b_i^{(k, \ell)}(x_1^{i-1}), x_i \right\rangle}{\sum_{k, \ell} q_{k, \ell} S_{i-1}(B^{(k, \ell)})} \]

\[ = \prod_{i=1}^{n} \frac{\sum_{k, \ell} q_{k, \ell} S_i(B^{(k, \ell)})}{\sum_{k, \ell} q_{k, \ell} S_{i-1}(B^{(k, \ell)})} \]
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\[ = \prod_{i=1}^{n} \frac{\sum_{k,\ell} q_{k,\ell} S_i(B^{(k,\ell)})}{\sum_{k,\ell} q_{k,\ell} S_{i-1}(B^{(k,\ell)})} \]

\[ = \sum_{k,\ell} q_{k,\ell} S_n(B^{(k,\ell)}), \]
The strategy $B = B^H$ then arises from weighing the elementary portfolio strategies $B^{(k,\ell)} = \{b_n^{(k,\ell)}\}$ such that the investor’s capital becomes

$$S_n(B) = \sum_{k,\ell} q_{k,\ell} S_n(B^{(k,\ell)}).$$
Assume that

\begin{enumerate}[(a)]
  \item the sequence of partitions is nested, that is, any cell of $P_{\ell+1}$ is a subset of a cell of $P_\ell$, $\ell = 1, 2, \ldots$;
  \item if $\text{diam}(A) = \sup_{x, y \in A} \|x - y\|$ denotes the diameter of a set, then for any sphere $S$ centered at the origin
    \[ \lim_{\ell \to \infty} \max_{j: A_{\ell,j} \cap S \neq \emptyset} \text{diam}(A_{\ell,j}) = 0. \]
\end{enumerate}

Then the portfolio scheme $B^H$ defined above is universally consistent with respect to the class of all ergodic processes such that $E\{|\ln X(j)| < \infty$, for $j = 1, 2, \ldots, d.$
www.szit.bme.hu/~gyorfi/histog.ps
Proof

We have to prove that

$$\liminf_{n \to \infty} W_n(B) = \liminf_{n \to \infty} \frac{1}{n} \ln S_n(B) \geq W^* \quad \text{a.s.}$$

W.l.o.g. we may assume $S_0 = 1$, so that

$$W_n(B) = \frac{1}{n} \ln S_n(B)$$
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\[ \geq \frac{1}{n} \ln \left( \sup_{k, \ell} q_{k, \ell} S_n(B^{(k, \ell)}) \right) \]

\[ = \frac{1}{n} \sup_{k, \ell} \left( \ln q_{k, \ell} + \ln S_n(B^{(k, \ell)}) \right) \]
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$$= \frac{1}{n} \sup_{k, \ell} \left( \ln q_{k, \ell} + \ln S_n(B^{(k, \ell)}) \right)$$

$$= \sup_{k, \ell} \left( W_n(B^{(k, \ell)}) + \frac{\ln q_{k, \ell}}{n} \right).$$
Thus

\[
\liminf_{n \to \infty} W_n(B) \geq \liminf_{n \to \infty} \sup_{k, \ell} \left( W_n(B^{(k, \ell)}) + \frac{\ln q_{k, \ell}}{n} \right)
\]
Thus

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\[ \geq \sup_{k, \ell} \liminf_{n \to \infty} \left( W_n(B^{(k, \ell)}) + \frac{\ln q_{k, \ell}}{n} \right) \]

\[ = \sup_{k, \ell} \lim_{k \to \infty} \lim_{l \to \infty} \varepsilon_{k, \ell} = W^* \]
Thus

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$$= \sup_{k, \ell} \liminf_{n \to \infty} W_n(B^{(k, \ell)})$$

$$= \sup_{k, \ell} \epsilon_{k, \ell}$$

Since the partitions $P_\ell$ are nested, we have that

$$\sup_{k, \ell} \epsilon_{k, \ell} = \lim_{k \to \infty} \lim_{l \to \infty} \epsilon_{k, \ell} = W^*.$$
Kernel regression estimate

Kernel function $K(x) \geq 0$

Bandwidth $h > 0$

$$m_n(x) = \frac{\sum_{i=1}^{n} Y_i K \left( \frac{x - X_i}{h} \right)}{\sum_{i=1}^{n} K \left( \frac{x - X_i}{h} \right)}$$

Naive (window) kernel function $K(x) = I_{\{|x| \leq 1\}}$

$$m_n(x) = \frac{\sum_{i=1}^{n} Y_i I_{\{|x - X_i| \leq h\}}}{\sum_{i=1}^{n} I_{\{|x - X_i| \leq h\}}}$$

Györfi

Machine learning and portfolio selections. II.
choose the radius $r_{k,\ell} > 0$ such that for any fixed $k$,

$$\lim_{\ell \to \infty} r_{k,\ell} = 0.$$
choose the radius $r_{k,\ell} > 0$ such that for any fixed $k$,

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for $n > k + 1$, define the expert $b^{(k,\ell)}$ by

$$b^{(k,\ell)}(x_1^{n-1}) = \arg\max_b \sum_{\{k < i < n: \|x_{i-k}^{i-1} - x_{n-k}^{n-1}\| \leq r_{k,\ell}\}} \ln \langle b, x_i \rangle,$$

if the sum is non-void, and $b_0 = (1/d, \ldots, 1/d)$ otherwise.
The kernel-based portfolio scheme is universally consistent with respect to the class of all ergodic processes such that
\[ E\{ \ln X(j) \} < \infty, \text{ for } j = 1, 2, \ldots, d. \]


www.szit.bme.hu/~gyorfi/kernel.pdf
The $k$-nearest neighbor (NN) regression estimate is given by:

$$m_n(x) = \sum_{i=1}^{n} W_{ni}(x; X_1, \ldots, X_n) Y_i.$$ 

$W_{ni}$ is $1/k$ if $X_i$ is one of the $k$ nearest neighbors of $x$ among $X_1, \ldots, X_n$, and $W_{ni}$ is 0 otherwise.
choose $p_\ell \in (0, 1)$ such that $\lim_{\ell \to \infty} p_\ell = 0$ for fixed positive integers $k, \ell$ ($n > k + \hat{\ell} + 1$) introduce the set of the $\hat{\ell} = \lfloor p_\ell n \rfloor$ nearest neighbor matches:

\[
\hat{J}^{(k, \ell)}_n = \{ i; k + 1 \leq i \leq n \text{ such that } x^{i-1}_{i-k} \text{ is among the } \hat{\ell} \text{ NNs of } x^{n-1}_{n-k} \text{ in } x^1, \ldots, x^{n-1}_{n-k} \}.
\]
choose \( p_\ell \in (0, 1) \) such that \( \lim_{\ell \to \infty} p_\ell = 0 \) for fixed positive integers \( k, \ell \) \( (n > k + \hat{\ell} + 1) \) introduce the set of the \( \hat{\ell} = \lfloor p_\ell n \rfloor \) nearest neighbor matches:

\[
\hat{J}_n^{(k, \ell)} = \{ i; k + 1 \leq i \leq n \text{ such that } x_{i-k}^{i-1} \text{ is among the } \hat{\ell} \text{ NNs of } x_{n-k}^{n-1} \}
\]

Define the portfolio vector by

\[
b^{(k, \ell)}(x_1^{n-1}) = \arg \max_b \sum_{i \in \hat{J}_n^{(k, \ell)}} \ln \langle b, x_i \rangle
\]

if the sum is non-void, and \( b_0 = (1/d, \ldots, 1/d) \) otherwise.
If for any vector $s = s_1^k$ the random variable

$$\|X_1^k - s\|$$

has continuous distribution function, then the nearest-neighbor portfolio scheme is universally consistent with respect to the class of all ergodic processes such that $E\{ |\ln X(j)| \} < \infty$, for $j = 1, 2, \ldots d$.

NN is robust, there is no scaling problem.
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NN is robust, there is no scaling problem.

empirical log-optimal:

\[ h^{(k,\ell)}(x_1^{n-1}) = \arg \max_b \sum_{i \in J_n} \ln \langle b, x_i \rangle \]
empirical log-optimal:

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Taylor expansion: \(\ln z \approx h(z) = z - 1 - \frac{1}{2}(z - 1)^2\)
semi-log-optimal portfolio

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Taylor expansion: \( \ln z \approx h(z) = z - 1 - \frac{1}{2}(z - 1)^2 \) empirical semi-log-optimal:

\[ \tilde{h}^{(k,\ell)}(x_1^{n-1}) = \arg \max_{b} \sum_{i \in J_n} h(\langle b, x_i \rangle) = \arg \max_{b} \{ \langle b, m \rangle - \langle b, Cb \rangle \} \]
Semi-log-optimal portfolio

empirical log-optimal:

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smaller computational complexity: quadratic programming
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smaller computational complexity: quadratic programming

Conditions of the model:

Assume that

- the assets are arbitrarily divisible,
- the assets are available in unbounded quantities at the current price at any given trading period,
- there are no transaction costs,
- the behavior of the market is not affected by the actions of the investor using the strategy under investigation.
At www.szit.bme.hu/~oti/portfolio there are two benchmark data set from NYSE:

- The first data set consists of daily data of 36 stocks with length 22 years.
- The second data set contains 23 stocks and has length 44 years.
At www.szit.bme.hu/~oti/portfolio there are two benchmark data set from NYSE:

- The first data set consists of daily data of 36 stocks with length 22 years.
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Our experiment is on the second data set.
Kernel based semi-log-optimal portfolio selection with $k = 1, \ldots, 5$ and $l = 1, \ldots, 10$

$$r^2_{k,l} = 0.0001 \cdot d \cdot k \cdot l,$$
Kernel based semi-log-optimal portfolio selection with 
k = 1, \ldots, 5 and \ell = 1, \ldots, 10

\[ r_{k,\ell}^2 = 0.0001 \cdot d \cdot k \cdot \ell, \]

AAY of kernel based semi-log-optimal portfolio is 222.7%
Experiments on average annual yields (AAY)

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\]

AAY of kernel based semi-log-optimal portfolio is 222.7\% double the capital.
MORRIS had the best AAY, 30.1\%.
the BCRP had average AAY 35.2\%.
The average annual yields of the individual experts.

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