Lab - Probabilistic Graphical Models

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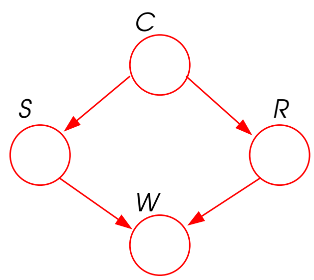
July 9, 2007

1. **Four random variables model**:

   - C: cloudy, $C \in \{0, 1\}$.
   - R: rain, $R \in \{0, 1\}$.
   - S: sprinkler on, $S \in \{0, 1\}$.
   - W: wet grass, $W \in \{0, 1\}$.



   (a) Write the joint probability of the 4 variables.

$$P(C, S, R, W) = P(C)P(S|C)P(R|C)P(W|R, S)$$

   (b) Given the following probability tables:

$$P(C = 1) = 0.5$$

$$P(S = 1|C = 1) = 0.1, \; P(S = 1|C = 0) = 0.5$$

$$P(R = 1|C = 1) = 0.8, \; P(R = 1|C = 0) = 0.2$$

$$P(W = 1|R = 1, S = 1) = 0.99$$

$$P(W = 1|R = 0, S = 1) = 0.9$$

$$P(W = 1|R = 1, S = 0) = 0.9$$

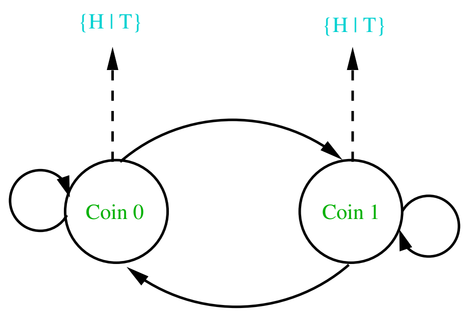
$$P(W = 1|R = 0, S = 0) = 0.0$$

   Compute $P(S = 1|W = 1)$ and $P(S = 1|W = 1, R = 1)$.

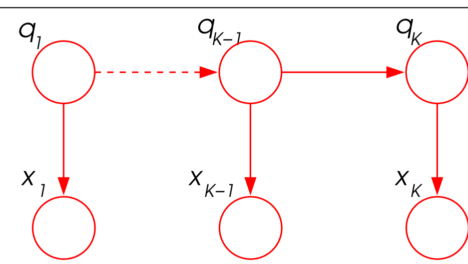
$$P(S=1|W=1) = \frac{P(W=1,S=1)}{P(W=1)}$$
$$= \frac{\sum_{R,C} P(W=1,S=1,R,C)}{\sum_{S,R,C} P(W=1,S,R,C)}$$
$$= \frac{\sum_{R,C} P(C)P(S=1|C)P(R|C)P(W=1|R,S=1)}{\sum_{S,R,C} P(C)P(S|C)P(R|C)P(W=1|R,S)}$$

$$P(S=1|W=1,R=1) = \frac{P(W=1,S=1,R=1)}{P(W=1,R=1)}$$
$$= \frac{\sum_{C} P(W=1,S=1,R=1,C)}{\sum_{S,C} P(W=1,S,R=1,C)}$$
$$= \frac{\sum_{C} P(C)P(S=1|C)P(R=1|C)P(W=1|R=1,S=1)}{\sum_{S,C} P(C)P(S|C)P(R=1|C)P(W=1|R=1,S)}$$

2. **A simple HMM.** Imagine that from the other side of a curtain I tell you that I have 2 biased coins C_1 and C_2, that following a Markov assumption I flip one or the other coin and that I give you the resulting sequence of heads and tails without telling you from which coin each component comes.



(a) Let $x_1^K = (x_1, \cdots, x_K)$ be a sequence of heads and tails, and $q_1^K = (q_1, \cdots, q_K)$ the corresponding sequence of coins. Draw the probabilistic graphical model associated with the joint distribution of x_1^K and q_1^K.



(b) What are the parameters of the model?

$p(q=1|q=2)$, $p(q=2|q=1)$, $p(q=1)$, $p(x=H|q=2)$, $p(x=H|q=1)$.

(c) Decompose the probability of having generated the sequence x_1^k and that the coin i have been flipped at time k, $\alpha(i,k) = p(x_1^k, q_k = i)$, in terms of $\alpha(j, k-1)$ and of the parameters.

$$\begin{aligned}
p(x_1^k, q_k = i) &= p(x_k | x_1^{k-1}, q_k = i) p(x_1^{k-1}, q_k = i) \\
&= p(x_k | q_k = i) \sum_j p(x_1^{k-1}, q_k = i, q_{k-1} = j) \\
&= p(x_k | q_k = i) \sum_j p(q_k = i | x_1^{k-1}, q_{k-1} = j) p(x_1^{k-1}, q_{k-1} = j) \\
&= p(x_k | q_k = i) \sum_j p(q_k = i | q_{k-1} = j) \alpha(j, k-1)
\end{aligned}$$

(d) How would you compute the likelihood $p(x_1^K)$ if you knew the parameters of the model? What would be the complexity of such calculation?

$$\begin{aligned}
p(x_1^K) &= \sum_i p(x_1^K, q_T = i) \\
&= \sum_i \alpha(i, T).
\end{aligned}$$

Complexity: $\mathcal{O}(N^2 T)$, where N is the number of states ($N = 2$ coins).