On Computational Thinking, Inferential Thinking and “Data Science”

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- "*It should only improve as we collect more data; in particular it shouldn't slow down*"
- "*There are serious privacy concerns of course, and they vary across the clients*"

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The Challenges are Daunting

- The core theories in computer science and statistics developed separately and there is an oil and water problem
- Core statistical theory doesn't have a place for runtime and other computational resources
- Core computational theory doesn't have a place for statistical risk

Outline

- Inference under privacy constraints
- Inference under communication constraints
- Lower bounds, the variational perspective and symplectic integration

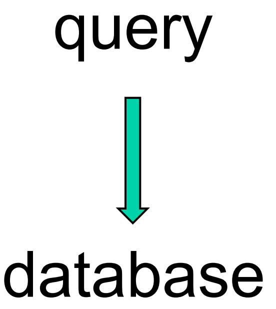
Part I: Inference and Privacy

with John Duchi and Martin Wainwright

Privacy and Data Analysis

- Individuals are not generally willing to allow their personal data to be used without control on how it will be used and now much privacy loss they will incur
- “Privacy loss” can be quantified via differential privacy
- We want to trade privacy loss against the value we obtain from “data analysis”
- The question becomes that of quantifying such value and juxtaposing it with privacy loss

Privacy



Privacy

query

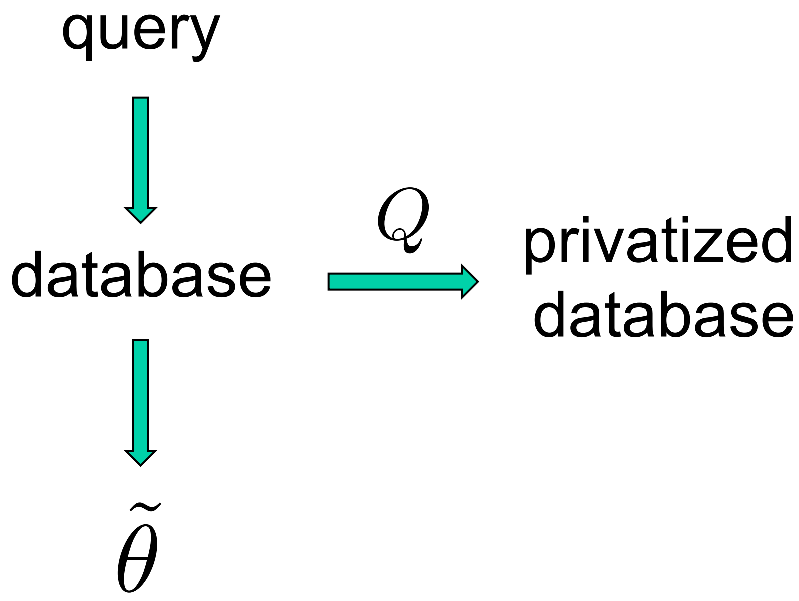
↓

database

↓

$$\tilde{\theta}$$

Privacy



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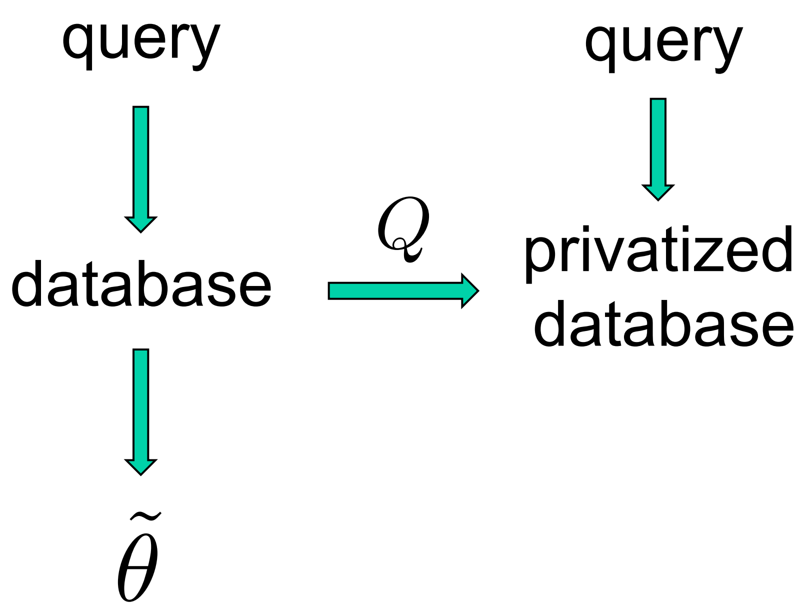

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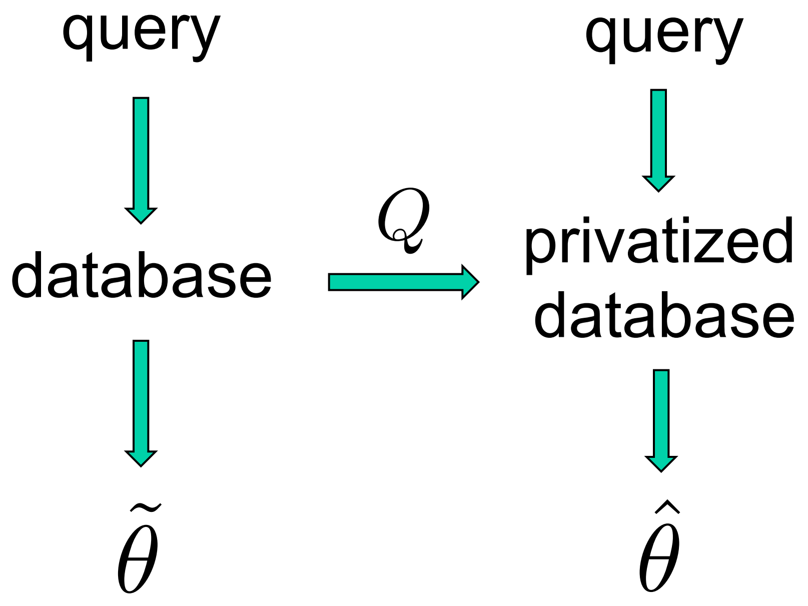
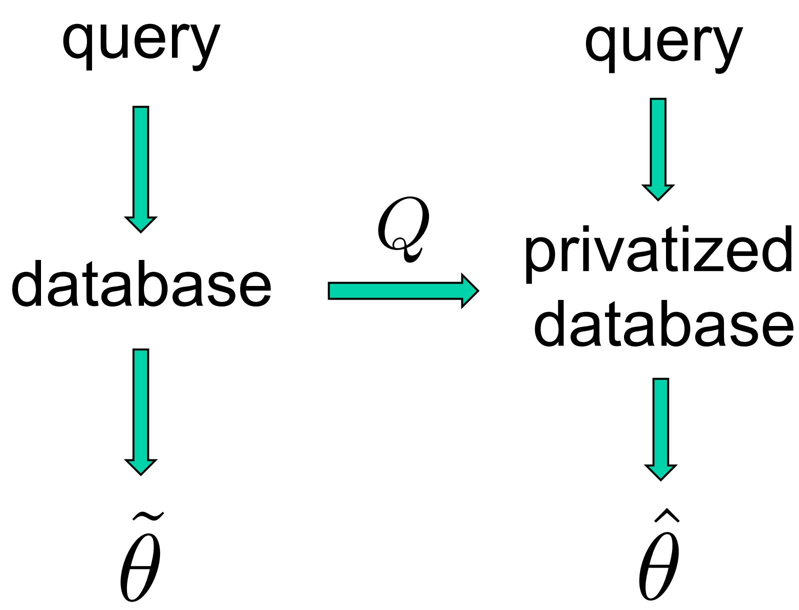

query
query
Q
database
privatized
database
$\tilde{\theta}$
$\hat{\theta}$

Privacy



Classical problem in differential privacy: show that $\hat{\theta}$ and $\tilde{\theta}$ are close under constraints on Q

Inference

query

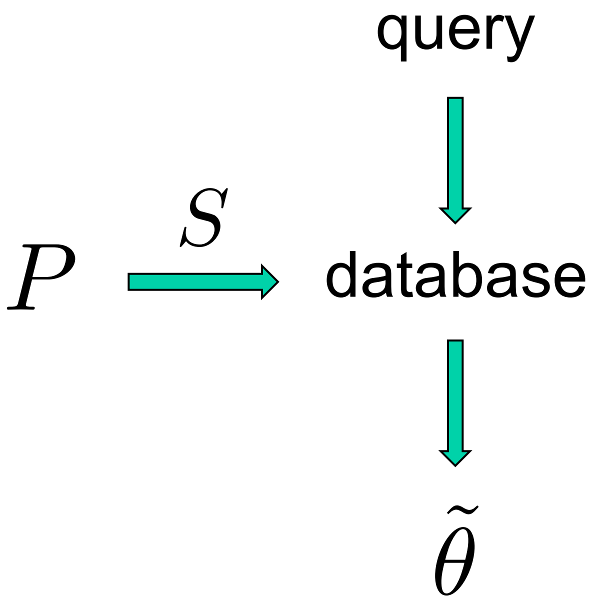
↓

database

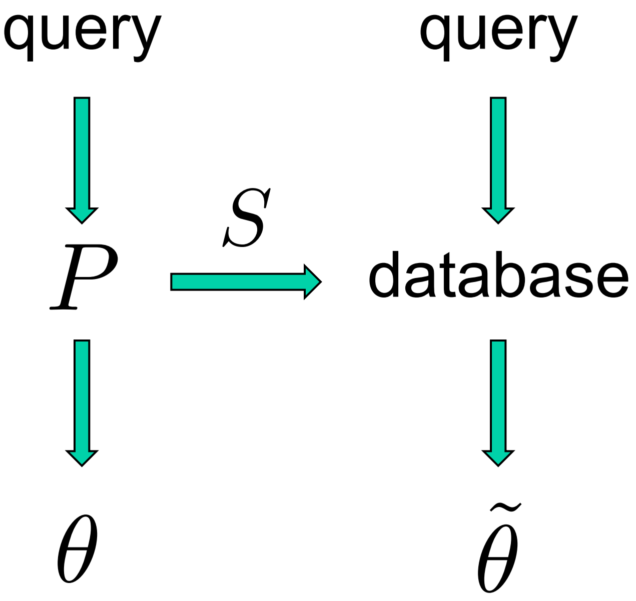
↓

$$\tilde{\theta}$$

Inference



Inference



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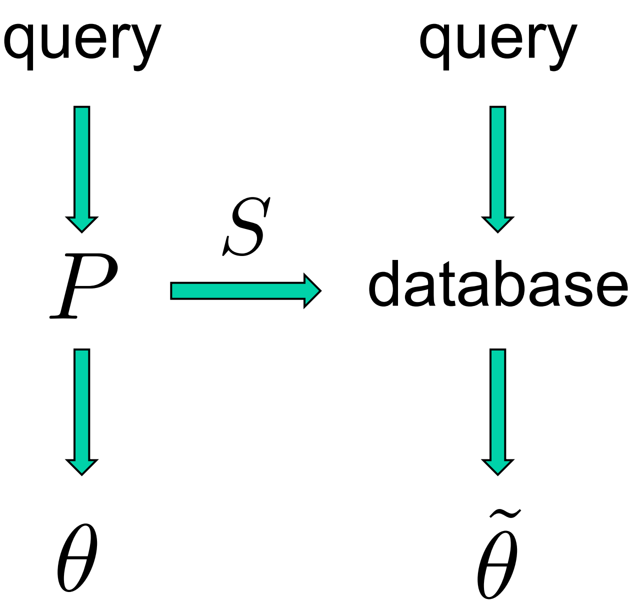
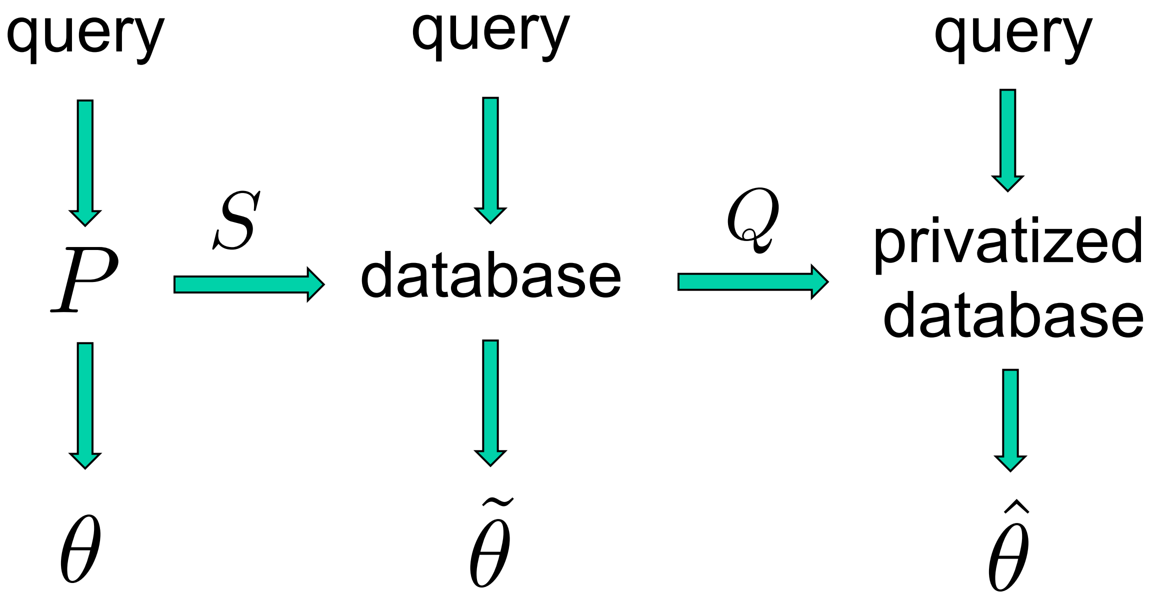


Classical problem in statistical theory: show that $\tilde{\theta}$ and θ are close under constraints on S

Privacy and Inference



The privacy-meets-inference problem: show that θ and $\hat{\theta}$ are close under constraints on Q and on S

Background on Inference

- In the 1930's, Wald laid the foundations of statistical decision theory
- Given a family of distributions $\mathcal{P}$, a parameter $\theta(P)$ for each $P \in \mathcal{P}$, an estimator $\hat{\theta}$, and a loss $l(\hat{\theta}, \theta(P))$, define the risk:

$$R_P(\hat{\theta}) := \mathbb{E}_P\left[l(\hat{\theta}, \theta(P))\right]$$

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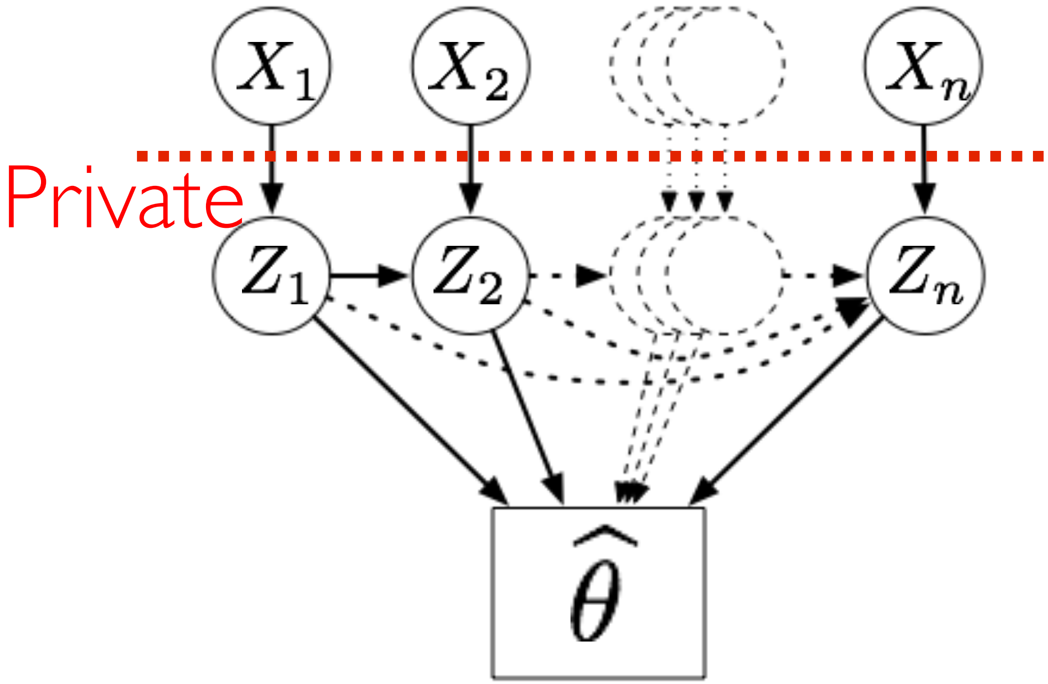
- Minimax principle [Wald, '39, '43]: choose estimator minimizing worst-case risk:

$$\sup_{P \in \mathcal{P}} \mathbb{E}_P\left[l(\hat{\theta}, \theta(P))\right]$$

Local Privacy



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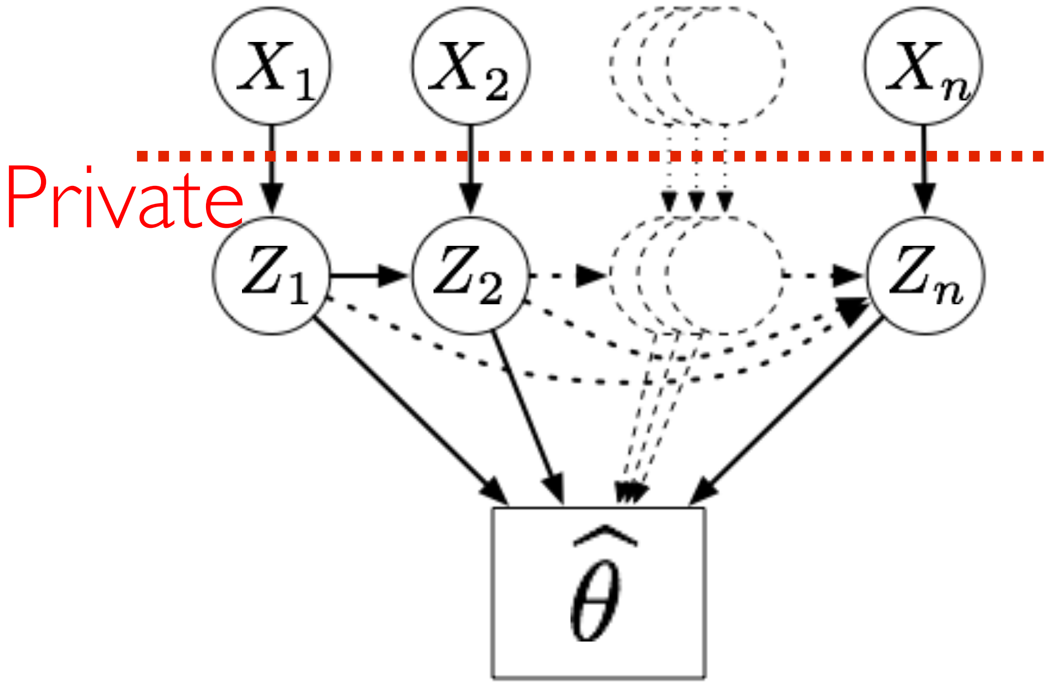
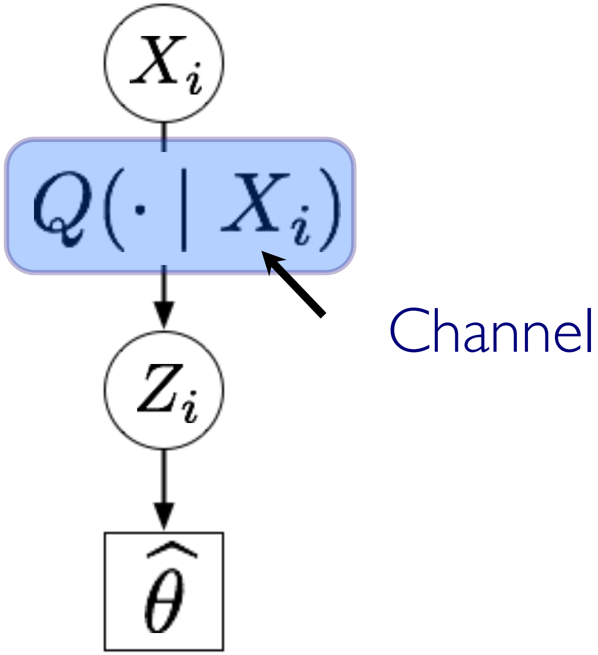




Individuals $i \in \{1, \ldots, n\}$ with private data $X_i \stackrel{\text{iid}}{\sim} P$

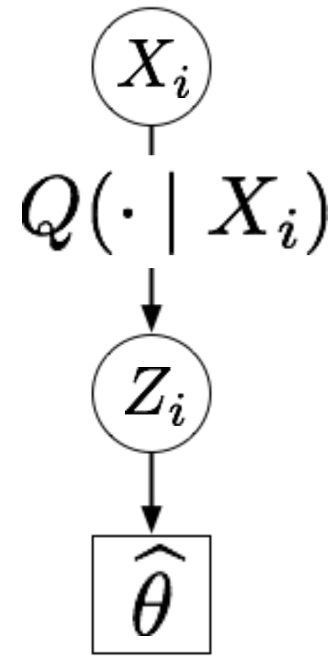
Estimator $Z_1^n \mapsto \widehat{\theta}(Z_1^n)$

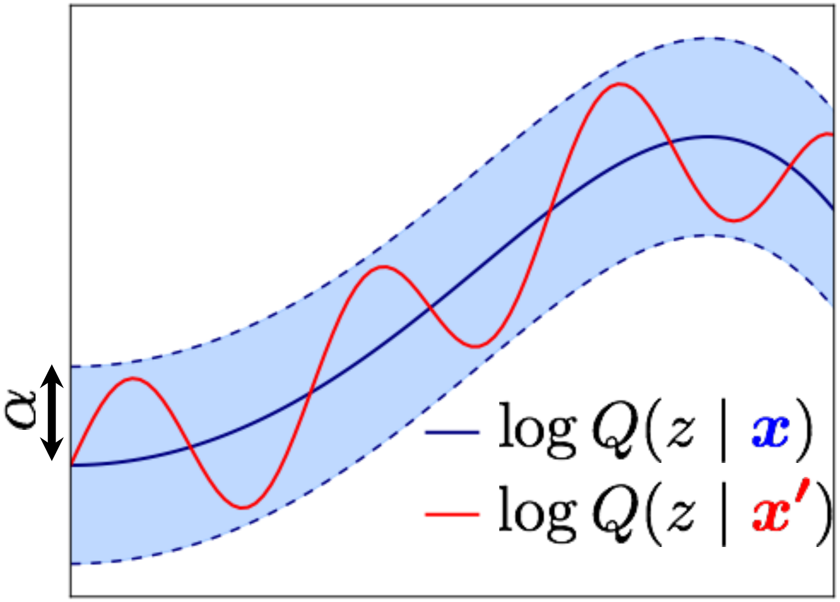
Differential Privacy

Definition: channel Q is α-differentially private if

$$\sup_{S, x \in \mathcal{X}, x' \in \mathcal{X}} \frac{Q(Z \in S \mid x)}{Q(Z \in S \mid x')} \leq \exp(\alpha)$$

[Dwork, McSherry, Nissim, Smith 06]





Given Z, cannot reliably discriminate between x and x'

Private Minimax Risk

- Parameter $\theta(P)$ of distribution
- Family of distributions $\mathcal{P}$
- Loss ℓ measuring error
- Family $\mathcal{Q}_\alpha$ of private channels

α -private Minimax risk

$$\mathfrak{M}_n(\theta(\mathcal{P}), \ell, \alpha) := \inf_{Q \in \mathcal{Q}_\alpha} \inf_{\widehat{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_{P,Q} \left[\ell(\widehat{\theta}(Z_1^n), \theta(P)) \right]$$

Best α-private channel

Minimax risk under privacy constraint

Vignette: Private Mean Estimation

Example: estimate reasons for hospital visits

Patients admitted to hospital for substance abuse

Estimate prevalence of different substances

- 1 Alcohol
- 1 Cocaine
- 0 Heroin
- 0 Cannabis
- 0 LSD
- 0 Amphetamines

Proportions $\boldsymbol{\theta} = \begin{array}{l} \theta_1 = .45 \\ \theta_2 = .32 \\ \theta_3 = .16 \\ \theta_4 = .20 \\ \theta_5 = .00 \\ \theta_6 = .02 \end{array}$

Vignette: Mean Estimation

Consider estimation of mean $\theta(P) := \mathbb{E}_P[X] \in \mathbb{R}^d$, with errors measured in ℓ_∞-norm, for

$$\mathcal{P}_d := \{\text{distributions } P \text{ supported on } [-1,1]^d\}$$

Proposition:

Minimax rate

$$\mathfrak{M}_n(\mathcal{P}_d, \|\cdot\|_\infty) \asymp \min\left\{1, \frac{\sqrt{\log d}}{\sqrt{n}}\right\}$$

(achieved by sample mean)

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Private minimax rate for $\alpha = O(1)$

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Note: Effective sample size $n \mapsto n\alpha^2/d$

Additional Examples

- Fixed-design regression
- Convex risk minimization
- Multinomial estimation
- Nonparametric density estimation

- Almost always, the effective sample size reduction is:

$$n \mapsto \frac{n\alpha^2}{d}$$

Computation and Inference

- How does inferential quality trade off against classical computational resources such as time and space?

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- Hard!

Computation and Inference: Mechanisms and Bounds

- Tradeoffs via convex relaxations
  - linking runtime to convex geometry and risk to convex geometry
- Tradeoffs via concurrency control
  - optimistic concurrency control
- Bounds via optimization oracles
  - number of accesses to a gradient as a surrogate for computation
- Bounds via communication complexity
- Tradeoffs via subsampling
  - bag of little bootstraps, variational consensus Monte Carlo

A Variational Framework for Accelerated Methods in Optimization

with Andre Wibisono and Ashia Wilson

July 12, 2016

Accelerated gradient descent

Setting: Unconstrained convex optimization

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- Accelerated gradient descent:

$$\begin{aligned} y_{k+1} &= x_k - \beta \nabla f(x_k) \\ x_{k+1} &= (1 - \lambda_k) y_{k+1} + \lambda_k y_k \end{aligned}$$

obtains the (optimal) convergence rate of $O(1/k^2)$

The acceleration phenomenon

Two classes of algorithms:

- **Gradient methods**
  - Gradient descent, mirror descent, cubic-regularized Newton's method (Nesterov and Polyak '06), etc.
  - Greedy descent methods, relatively well-understood

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  - Greedy descent methods, relatively well-understood
- **Accelerated methods**
  - Nesterov's accelerated gradient descent, accelerated mirror descent, accelerated cubic-regularized Newton's method (Nesterov '08), etc.
  - Important for both theory (optimal rate for first-order methods) and practice (many extensions: FISTA, stochastic setting, etc.)
  - *Not* descent methods, faster than gradient methods, still mysterious

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  - Optimized first-order method (Drori, Teboulle '14; Kim, Fessler '15)
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  But only for strongly convex functions, or first-order methods

Question: What is the underlying mechanism that generates acceleration (including for higher-order methods)?

Accelerated methods: Continuous time perspective

- Gradient descent is discretization of gradient flow

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- These ODEs are obtained by taking continuous time limits. Is there a deeper generative mechanism?

Our work: A general variational approach to acceleration

A systematic discretization methodology

Bregman Lagrangian

Define the **Bregman Lagrangian**:

$$\mathcal{L}(x, \dot{x}, t) = e^{\gamma_t + \alpha_t} \left(D_h(x + e^{-\alpha_t} \dot{x}, x) - e^{\beta_t} f(x) \right)$$

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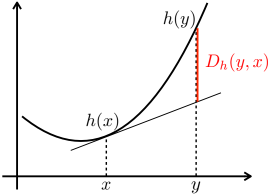
- $D_h(y, x) = h(y) - h(x) - \langle \nabla h(x), y - x \rangle$ is the Bregman divergence
- h is the convex distance-generating function



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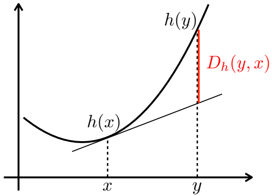

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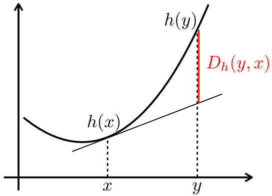

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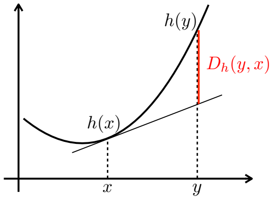

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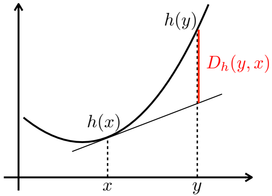

Ideal scaling conditions:

$$\dot{\beta}_t \leq e^{\alpha_t}$$
$$\dot{\gamma}_t = e^{\alpha_t}$$

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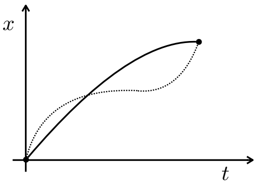
Variational problem over curves:

$$\min_X \int \mathcal{L}(X_t, \dot{X}_t, t)\, dt$$



Optimal curve is characterized by **Euler-Lagrange** equation:

$$\frac{d}{dt}\left\{ \frac{\partial \mathcal{L}}{\partial \dot{x}}(X_t, \dot{X}_t, t) \right\} = \frac{\partial \mathcal{L}}{\partial x}(X_t, \dot{X}_t, t)$$

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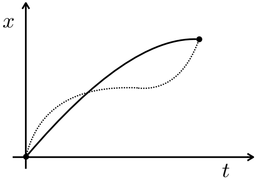

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E-L equation for Bregman Lagrangian under ideal scaling:

$$\ddot{X}_t + (e^{\alpha_t} - \dot{\alpha}_t)\dot{X}_t + e^{2\alpha_t + \beta_t} \left[\nabla^2 h(X_t + e^{-\alpha_t}\dot{X}_t) \right]^{-1} \nabla f(X_t) = 0$$

General convergence rate

Theorem

Theorem Under ideal scaling, the E-L equation has convergence rate

$$f(X_t) - f(x^*) \leq O(e^{-\beta_t})$$

General convergence rate

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Proof. Exhibit a Lyapunov function for the dynamics:

$$\mathcal{E}_t = D_h\left(x^*,\, X_t + e^{-\alpha_t}\dot{X}_t\right) + e^{\beta_t}(f(X_t) - f(x^*))$$

$$\dot{\mathcal{E}}_t = -e^{\alpha_t+\beta_t}D_f(x^*, X_t) + (\dot{\beta}_t - e^{\alpha_t})e^{\beta_t}(f(X_t) - f(x^*)) \;\leq\; 0$$

□

Note: Only requires convexity and differentiability of f, h

Polynomial convergence rate

For $p > 0$, choose parameters:

$$\alpha_t = \log p - \log t$$
$$\beta_t = p \log t + \log C$$
$$\gamma_t = p \log t$$

E-L equation has $O(e^{-\beta_t}) = O(1/t^p)$ convergence rate:

$$\ddot{X}_t + \frac{p+1}{t}\dot{X}_t + Cp^2 t^{p-2}\Big[\nabla^2 h\Big(X_t + \frac{t}{p}\dot{X}_t\Big)\Big]^{-1}\nabla f(X_t) = 0$$

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E-L equation has $O(e^{-\beta_t}) = O(1/t^p)$ convergence rate:

$$\ddot{X}_t + \frac{p+1}{t}\dot{X}_t + Cp^2 t^{p-2}\left[\nabla^2 h\Big(X_t + \frac{t}{p}\dot{X}_t\Big)\right]^{-1}\nabla f(X_t) = 0$$

For $p = 2$:

- Recover result of Krichene et al with $O(1/t^2)$ convergence rate
- In Euclidean case, recover ODE of Su et al: $\ddot{X}_t + \frac{3}{t}\dot{X}_t + \nabla f(X_t) = 0$

Time dilation property (reparameterizing time)

($p = 2$: accelerated gradient descent)

$$O\left(\frac{1}{t^2}\right): \quad \ddot{X}_t + \frac{3}{t}\dot{X}_t + 4C\Big[\nabla^2 h\Big(X_t + \frac{t}{2}\dot{X}_t\Big)\Big]^{-1}\nabla f(X_t) = 0$$

$$\Big\downarrow \quad \text{speed up time: } Y_t = X_{t^{3/2}}$$

$$O\left(\frac{1}{t^3}\right): \quad \ddot{Y}_t + \frac{4}{t}\dot{Y}_t + 9Ct\Big[\nabla^2 h\Big(Y_t + \frac{t}{3}\dot{Y}_t\Big)\Big]^{-1}\nabla f(Y_t) = 0$$

($p = 3$: accelerated cubic-regularized Newton's method)

Time dilation property (reparameterizing time)

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($p = 3$: accelerated cubic-regularized Newton's method)

- All accelerated methods are traveling the same curve in space-time at different speeds
- Gradient methods don't have this property
  - From gradient flow to rescaled gradient flow: Replace $\frac{1}{2}\|\cdot\|^2$ by $\frac{1}{p}\|\cdot\|^p$

Time dilation for general Bregman Lagrangian

$$O(e^{-\beta_t}): \quad \text{E-L for Lagrangian } \mathcal{L}_{\alpha,\beta,\gamma}$$

$$\Big\downarrow \quad \text{speed up time: } Y_t = X_{\tau(t)}$$

$$O(e^{-\beta_{\tau(t)}}): \quad \text{E-L for Lagrangian } \mathcal{L}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$$

where

$$\tilde{\alpha}_t = \alpha_{\tau(t)} + \log \dot{\tau}(t)$$
$$\tilde{\beta}_t = \beta_{\tau(t)}$$
$$\tilde{\gamma}_t = \gamma_{\tau(t)}$$

Time dilation for general Bregman Lagrangian

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where

$$\begin{aligned} \tilde{\alpha}_t &= \alpha_{\tau(t)} + \log \dot{\tau}(t) \\ \tilde{\beta}_t &= \beta_{\tau(t)} \\ \tilde{\gamma}_t &= \gamma_{\tau(t)} \end{aligned}$$

Question: How to discretize E-L while preserving the convergence rate?

Discretizing the dynamics (naive approach)

Write E-L as a system of first-order equations:

$$Z_t = X_t + \frac{t}{p}\dot{X}_t$$

$$\frac{d}{dt}\nabla h(Z_t) = -Cpt^{p-1}\nabla f(X_t)$$

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Euler discretization with time step $\delta > 0$ (i.e., set $x_k = X_t$, $x_{k+1} = X_{t+\delta}$):

$$x_{k+1} = \frac{p}{k+p}z_k + \frac{k}{k+p}x_k$$
$$z_k = \arg\min_z \left\{ Cpk^{(p-1)}\langle \nabla f(x_k), z\rangle + \frac{1}{\epsilon}D_h(z, z_{k-1}) \right\}$$

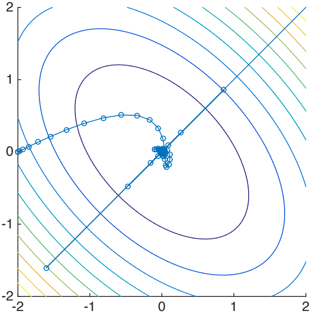
with step size $\epsilon = \delta^p$, and $k^{(p-1)} = k(k+1)\cdots(k+p-2)$ is the rising factorial

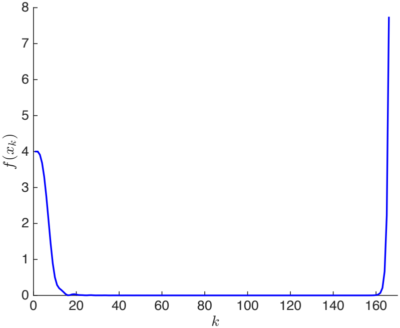
Naive discretization doesn't work

$$x_{k+1} = \frac{p}{k+p} z_k + \frac{k}{k+p} x_k$$

$$z_k = \arg\min_z \left\{ Cpk^{(p-1)} \langle \nabla f(x_k), z \rangle + \frac{1}{\epsilon} D_h(z, z_{k-1}) \right\}$$

Cannot obtain a convergence guarantee, and empirically unstable





Modified discretization

Introduce an auxiliary sequence y_k:

$$x_{k+1} = \frac{p}{k+p} z_k + \frac{k}{k+p} y_k$$

$$z_k = \arg\min_z \left\{ Cpk^{(p-1)} \langle \nabla f(y_k), z \rangle + \frac{1}{\epsilon} D_h(z, z_{k-1}) \right\}$$

Sufficient condition: $\langle \nabla f(y_k), x_k - y_k \rangle \geq M \epsilon^{\frac{1}{p-1}} \|\nabla f(y_k)\|_*^{\frac{p}{p-1}}$

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Assume h is uniformly convex: $D_h(y, x) \geq \frac{1}{p} \|y - x\|^p$

Theorem

Theorem

$$f(y_k) - f(x^*) \leq O\left(\frac{1}{\epsilon k^p}\right)$$

Note: Matching convergence rates $1/(\epsilon k^p) = 1/(\delta k)^p = 1/t^p$

Proof using generalization of Nesterov's estimate sequence technique

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Assume h is uniformly convex: $D_h(y, x) \geq \frac{1}{p} \|y - x\|^p$

Theorem

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Proof using generalization of Nesterov's estimate sequence technique

Higher-order gradient update

Higher-order Taylor approximation of f:

$$f_{p-1}(y; x) = f(x) + \langle \nabla f(x), y - x \rangle + \cdots + \frac{1}{(p-1)!} \nabla^{p-1} f(x)(y - x)^{p-}$$

Higher-order gradient update:

$$y_k = \arg\min_y \left\{ f_{p-1}(y; x_k) + \frac{2}{\epsilon p} \|y - x_k\|^p \right\}$$

Higher-order gradient update

Higher-order Taylor approximation of f:

$$f_{p-1}(y;x) \;=\; f(x) + \langle \nabla f(x), y - x \rangle + \cdots + \frac{1}{(p-1)!} \nabla^{p-1} f(x)(y-x)^{p-}$$

Higher-order gradient update:

$$y_k \;=\; \arg\min_y \left\{ f_{p-1}(y; x_k) + \frac{2}{\epsilon p} \|y - x_k\|^p \right\}$$

Assume f is smooth of order $p-1$:

$$\|\nabla^{p-1} f(y) - \nabla^{p-1} f(x)\|_* \le \frac{1}{\epsilon} \|y - x\|$$

Theorem

Lemma

$$\langle \nabla f(y_k), x_k - y_k \rangle \;\ge\; \frac{1}{4} \epsilon^{\frac{1}{p-1}} \|\nabla f(y_k)\|_*^{\frac{p}{p-1}}$$

Can use this to complete the modified discretization process!

Accelerated higher-order gradient method

$$x_{k+1} = \frac{p}{k+p} z_k + \frac{k}{k+p} y_k$$

$$y_k = \arg\min_y \left\{ f_{p-1}(y; x_k) + \frac{2}{\epsilon p} \|y - x_k\|^p \right\} \quad \leftarrow O\left(\frac{1}{\epsilon k^{p-1}}\right)$$

$$z_k = \arg\min_z \left\{ Cpk^{(p-1)} \langle \nabla f(y_k), z \rangle + \frac{1}{\epsilon} D_h(z, z_{k-1}) \right\}$$

If $\nabla^{p-1} f$ is $(1/\epsilon)$-Lipschitz and h is uniformly convex of order p, then:

$$f(y_k) - f(x^*) \leq O\left(\frac{1}{\epsilon k^p}\right) \quad \leftarrow \text{ accelerated rate}$$

$p = 2$: Accelerated gradient/mirror descent

$p = 3$: Accelerated cubic-regularized Newton's method (Nesterov '08)

$p \geq 2$: Accelerated higher-order method

Recap: Gradient vs. accelerated methods

How to design dynamics for minimizing a convex function f?

Rescaled gradient flow

$$\dot{X}_t = -\nabla f(X_t) \,/\, \|\nabla f(X_t)\|_*^{\frac{p-2}{p-1}}$$

$$O\left(\frac{1}{t^{p-1}}\right)$$

Higher-order gradient method

$O\left(\frac{1}{\epsilon k^{p-1}}\right)$ when $\nabla^{p-1} f$ is $\frac{1}{\epsilon}$-Lipschitz

matching rate with $\epsilon = \delta^{p-1} \Leftrightarrow \delta = \epsilon^{\frac{1}{p-1}}$

Recap: Gradient vs. accelerated methods

How to design dynamics for minimizing a convex function f?

Rescaled gradient flow	Polynomial Euler-Lagrange equation
$\dot{X}_t = -\nabla f(X_t) \,/\, \|\nabla f(X_t)\|_*^{\frac{p-2}{p-1}}$	$\ddot{X}_t + \frac{p+1}{t}\dot{X}_t + t^{p-2}\left[\nabla^2 h\left(X_t + \frac{t}{p}\dot{X}_t\right)\right]^{-1}\nabla f(X_t) =$
$O\left(\frac{1}{t^{p-1}}\right)$	$O\left(\frac{1}{t^p}\right)$
Higher-order gradient method	**Accelerated higher-order method**
$O\left(\frac{1}{\epsilon k^{p-1}}\right)$ when $\nabla^{p-1} f$ is $\frac{1}{\epsilon}$-Lipschitz	$O\left(\frac{1}{\epsilon k^{p}}\right)$ when $\nabla^{p-1} f$ is $\frac{1}{\epsilon}$-Lipschitz
matching rate with $\epsilon = \delta^{p-1} \Leftrightarrow \delta = \epsilon^{\frac{1}{p-1}}$	matching rate with $\epsilon = \delta^{p} \Leftrightarrow \delta = \epsilon^{\frac{1}{p}}$

Summary: Bregman Lagrangian

- Bregman Lagrangian family with general convergence guarantee

$$\mathcal{L}(x, \dot{x}, t) = e^{\gamma_t + \alpha_t} \left(D_h(x + e^{-\alpha_t}\dot{x}, x) - e^{\beta_t} f(x) \right)$$

- Polynomial subfamily generates accelerated higher-order methods: $O(1/t^p)$ convergence rate via higher-order smoothness

Summary: Bregman Lagrangian

- Bregman Lagrangian family with general convergence guarantee

$$\mathcal{L}(x, \dot{x}, t) = e^{\gamma_t + \alpha_t} \left(D_h(x + e^{-\alpha_t}\dot{x}, x) - e^{\beta_t} f(x) \right)$$

- Polynomial subfamily generates accelerated higher-order methods: $O(1/t^p)$ convergence rate via higher-order smoothness
- Exponential subfamily: $O(e^{-ct})$ rate via uniform convexity
- Understand structure and properties of Bregman Lagrangian: Gauge invariance, symmetry, gradient flows as limit points, etc.
- Bregman Hamiltonian:

$$\mathcal{H}(x, p, t) = e^{\alpha_t + \gamma_t} \left(D_{h^*}\left(\nabla h(x) + e^{-\gamma_t} p,\ \nabla h(x)\right) + e^{\beta_t} f(x) \right)$$