Complexity of Discrete Energy Minimization Problems

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Energy Minimization

- Used in many CV applications

3D Room Layout
[Schwing and Urtasun, 2012]
Energy Minimization

- Used in many CV applications

Semantic Segmentation
[Ren et al, 2012]
Energy Minimization

- Used in many CV applications

Shape and Image In-painting
[Shekhovtsov et al, 2012]
Energy Minimization

- Used in many CV applications
- Pairwise and higher order

$$\min_{x \in \mathcal{L}^V} \sum_{u \in V} f_u(x_u) + \sum_{(u,v) \in \mathcal{E}} f_{uv}(x_u, x_v)$$
- But is it efficient?

Shape and Image In-painting
[Shekhovtsov et al, 2012]
Energy Minimization

- Used in many CV applications
- Pairwise and higher order
  \[
  \min_{x \in \mathcal{X}} \sum_{u \in \mathcal{V}} f_u(x_u) + \sum_{(u,v) \in \mathcal{E}} f_{uv}(x_u, x_v)
  \]
- But is it efficient?
- In general, NP-hard
- There are tractable classes

Shape and Image In-painting
[Shekhovtsov et al, 2012]

Bounded Treewidth
Binary Outerplanar
Convex Interaction
Submodular

P Optimization

NP-hard

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Energy Minimization

- Used in many CV applications
- Pairwise and higher order
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- But is it efficient?
- In general, NP-hard
- There are tractable classes
- Growing corpus of general “approximate” inference methods
- Learning Project with QPBO
Energy Minimization

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$$\min_{x \in \mathcal{L}^\mathcal{V}} \sum_{u \in \mathcal{V}} f_u(x_u) + \sum_{(u,v) \in \mathcal{E}} f_{uv}(x_u, x_v)$$

Quadratic Pseudo-Boolean Optimization (QPBO):

$$\min_{x \in \{0,1\}^\mathcal{V}} \sum_{u \in \mathcal{V}} a_u x_u + \sum_{(u,v) \in \mathcal{E}} a_{uv} x_u x_v$$

- Learning Project with QPBO
  - Difficult instances during learning
Energy Minimization

- Used in many CV applications
- Pairwise and higher order
  \[ \min_{x \in \mathcal{L}^V} \sum_{u \in V} f_u(x_u) + \sum_{(u,v) \in \mathcal{E}} f_{uv}(x_u, x_v) \]
- But is it efficient?
- In general, NP-hard
- There are tractable classes
- Growing corpus of general “approximate” inference methods
- Learning Project with QPBO
  - Difficult instances during learning
  - Could benefit from approximation guarantees
- **We find QPBO and general energy minimization to be inapproximable**
NP-hard Problems Vary Greatly in Approximability

- Approximation Ratio: $f(x)/f(x^*), f(x^*) > 0$

- Classes of approximation:
  - PTAS - ratio $1+\varepsilon$ in polynomial time (knapsack, Euclidean TSP)
  - APX - constant approximation ratio
  - log-APX - logarithmic in bit-length
  - poly-APX - polynomial

- APX / log-APX indicate more practical classes
  - Algorithms can build on achieving guarantees
  - Much better ratios per instance

- APX-hard is not too bad! How hard is QPBO?

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Which Energy Problems are not in APX?

- class exp-APX:
  - Approximation ratio exponential in bit length
  - Suffices to find any feasible solution

Theorem: QPBO (energy with 2 labels) is complete in exp-APX
- Any problem from exp-APX can be reduced to QPBO
- In polynomial time
- While preserving approximation ratio

Theorem: Planar energy with 3+ labels is complete in exp-APX

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Proof Scheme

Turing Machine

- AP-reduction
  - [Opronen & Mannila 09]

Weighted 3-SAT

- AP-reduction (using Ishikawa’s reduction)

QPBO

- AP-reduction

Planar Energy with 3 labels

- Weighted 3-SAT
  - QPBO
  - General Energy Minimization
    - Planar with 3+ labels

NP-hard

- exp-APX
  - QPBO

poly-APX

- log-APX
  - Metric Labeling
  - Potts Model
  - Truncated Linear
  - MAX-CUT
  - Planar Vertex Cover
  - Convex Interaction
  - Binary Outerplanar
  - Submodular
  - Bounded Treewidth

P Optimization
Take Away Message

- Energy minimization problems vary in approximation ratio
- Bounded approximation ratio
  - Indicates a class of practical interest
  - Useful for algorithm design (Primal-Dual)
- Do not try to prove approximation guarantee if
  - Model includes QPBO/planar 3-label/general energy minimization
  - Or you can build AP-reduction from them